Finite state transducers for modular Möbius number systems

Martin Delacourt[1] and Petr Kůrka[2]

[1] Laboratoire d'Informatique Fondamentale de Marseille, 39 rue Joliot Curie, F-13453 Marseille Cedex, France.
[2] Center for Theoretical Study, Academy of Sciences and Charles University in Prague, Jilská 1, CZ-11000 Praha 1, Czechia.

Abstract. Modular Möbius number systems consist of Möbius transformations with integer coefficients and unit determinant. We show that in any modular Möbius number system, the computation of a Möbius transformation with integer coefficients can be performed by a finite state transducer and has linear time complexity. As a byproduct we show that every modular Möbius number system has the expansion subshift of finite type.

Keywords: exact real algorithms, expansion subshift, absorptions, emissions.

1 Introduction

In an unpublished but influential manuscript, Gosper [1] shows that continued fractions can be used for arithmetical algorithms, provided they are redundant. Based on these ideas, **exact real arithmetical algorithms** have been developed in Vuillemin [15], Kornerup and Matula [4] or Potts [13]. These algorithms perform a sequence of **input absorptions** and **output emissions** and update their inner state, which may be a (2×2)-matrix in the case of a Möbius transformation or a (2×4)-matrix in the case of binary operations like addition or multiplication.

Using the concepts and methods of symbolic dynamics, exact real arithmetic has been generalized in the theory of **Möbius number systems** (MNS) introduced in Kůrka [6] and developed in Kůrka and Kazda [10]. Möbius number systems represent real numbers by infinite words from a one-sided **expansion subshift**. The letters of the alphabet stand for real orientation-preserving Möbius transformations and the concatenation of letters corresponds to the composition of transformations. In Kůrka [7] we have investigated MNS in which rational numbers have periodic or preperiodic expansions and in Kůrka [9] we have characterized MNS whose expansion subshifts are of finite type or sofic.

The time complexity of the unary exact real algorithm which computes a Möbius transformation depends on the growth of its inner state matrices during the computation. Heckmann [2] analyzes this process in positional number systems and proves the **Law of big numbers** (not to be confused with the Law of large numbers), saying that the norm of the state matrix after n absorptions or emissions is at least of the order $r^{n/2}$ for r-ary positional systems. This implies that the bit size of the state matrices grows at least linearly, and arithmetical operations have quadratic time complexity. In Kůrka [8] we have shown that in a general MNS the growth of the state matrices can be slower and we conjectured that the state matrices can even remain bounded. In the present paper we show that this is the case for modular MNS, i.e., MNS whose transformations have integer coefficients and unit determinant. It follows that the unary algorithm can be realized by a finite state transducer and has linear time complexity. This generalizes the results of Raney [14] and complements the results of Konečný [3], who proves (in a slightly different context), that the only differentiable functions computable by finite state transducers are Möbius transformations.

2 Möbius transformations

The **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ can be regarded as a projective space, i.e., the space of one-dimensional subspaces of the two-dimensional vector space. On $\overline{\mathbb{R}}$ we have **homogeneous coordinates** $x = (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with equality $x = y$ iff $\det(x, y) = x_0 y_1 - x_1 y_0 = 0$. We regard $x \in \overline{\mathbb{R}}$ as a column vector, and write it usually as $x = \frac{x_0}{x_1} = x_0/x_1$, for example $\infty = 1/0$. The **stereographic projection** $\mathbf{h}(z) = (iz+1)/(z+i)$ maps $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : \ |z| = 1\}$ in the complex plane, and the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$ conformally to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$.

A real **orientation-preserving Möbius transformation** (MT) is a self-map of $\overline{\mathbb{R}}$ of the form

$$M_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d} = \frac{ax_0 + bx_1}{cx_0 + dx_1},$$

where $a, b, c, d \in \mathbb{R}$ and $\det(M_{(a,b,c,d)}) = ad - bc > 0$. Möbius transformations form a group and act also on the upper half-plane $\mathbb{U}$: If $z \in \mathbb{U}$ then $M(z) \in \mathbb{U}$ as well. On $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ we get **disc Möbius transformations** defined by $\widehat{M}_{(a,b,c,d)}(z) = \mathbf{h} \circ M_{(a,b,c,d)} \circ \mathbf{h}^{-1}(z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$, where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. The **circle derivation** of $M = M_{(a,b,c,d)}$ at $x \in \overline{\mathbb{R}}$ is defined by

$$M^\bullet(x) = |\widehat{M}'(\mathbf{h}(x))| = \frac{(ad-bc) \cdot (x_0^2 + x_1^2)}{(ax_0 + bx_1)^2 + (cx_0 + dx_1)^2} = \frac{\det(M) \cdot ||x||^2}{||M(x)||^2}.$$

The **expansion interval** of an MT is $\mathbf{V}(M) = \{x \in \overline{\mathbb{R}} : \ (M^{-1})^\bullet(x) > 1\}$. If $M = R_\alpha = M_{(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, -\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2})}$ is a rotation, then $M^\bullet(x) = 1$ and $\mathbf{V}(M)$ is empty. Otherwise $\mathbf{V}(M)$ is a proper set interval.

3 Intervals

A **set interval** is an open connected subset of $\overline{\mathbb{R}}$. A **proper set interval** is a nonempty set interval properly included in $\overline{\mathbb{R}}$. We represent proper set intervals by (2×2)-matrices whose columns are their left and right endpoints. The stereographic projection applied to $x = \frac{r\sin\alpha}{r\cos\alpha} \in \overline{\mathbb{R}}$ gives $\mathbf{h}(x) = \sin 2\alpha - i\cos 2\alpha = e^{i(2\alpha-\frac{\pi}{2})}$, so it doubles the angles. Matrices with columns $x = \frac{r\sin\alpha}{r\cos\alpha}$, $y = \frac{s\sin\beta}{s\cos\beta}$ where $0 \le \alpha < 2\pi$, $\alpha < \beta < \alpha+\pi$ therefore represent all proper intervals. Since $\det(x,y) = rs\sin(\alpha-\beta) < 0$, we define matrix intervals as (2×2)-matrices with negative determinant and write them as pairs $I = (\frac{x_0}{x_1}, \frac{y_0}{y_1})$ of their left and right endpoints $\mathfrak{l}(I) = \frac{x_0}{x_1}$, $\mathfrak{r}(I) = \frac{y_0}{y_1}$. The set of **matrix intervals** is therefore

$$\mathbb{I}(\mathbb{R}) = \{(\tfrac{x_0}{x_1}, \tfrac{y_0}{y_1}) \in \mathrm{GL}(\mathbb{R},2) :\ x_0y_1 - x_1y_0 < 0\}.$$

We define the **size** and the **length** of an interval (x,y) by

$$\begin{aligned}\mathrm{sz}(x,y) &= \frac{x_0y_0 + x_1y_1}{x_0y_1 - x_1y_0} = \frac{x\cdot y}{\det(x,y)},\\ |(x,y)| &= \frac{1}{2} + \frac{1}{\pi}\arctan\mathrm{sz}(x,y).\end{aligned}$$

For $x = \frac{r\sin\alpha}{r\cos\alpha}$, $y = \frac{s\sin\beta}{s\cos\beta}$ we get $\mathrm{sz}(x,y) = -\cot(\beta-\alpha) = \tan(\beta-\alpha-\frac{\pi}{2})$, so $|(x,y)| = (\beta-\alpha)/\pi$, provided $0 < \beta-\alpha < \pi$. The length $|I| \in (0,1)$ of I is an increasing function of the size $\mathrm{sz}(I) \in (-\infty,+\infty)$ of I. A matrix interval $I = (x,y)$ defines an open set interval by $z \in I \iff \det(x,z)\cdot\det(z,y) > 0$, and a closed set interval $z \in \overline{I} \iff \det(x,z)\cdot\det(z,y) \ge 0$. If $I = (\frac{r\sin\alpha}{r\cos\alpha}, \frac{s\sin\beta}{s\cos\beta})$, then $z = \frac{t\sin\gamma}{t\cos\gamma} \in I$ iff either $\alpha < \gamma < \beta$ or $\alpha+\pi < \gamma < \beta+\pi$. If I, J are intervals, then $I \subseteq J$ iff $\mathfrak{l}(I) \in \overline{J}$ and $\mathfrak{r}(I) \in \overline{J}$. In this case $\mathrm{sz}(I) \le \mathrm{sz}(J)$. When we transform intervals, we work with the matrix representations of MT rather than with the transformations themselves. Möbius transformations are represented by matrices

$$\mathbb{M}(\mathbb{R}) = \{M_{(a,b,c,d)} \in \mathrm{GL}(\mathbb{R},2) :\ ad - bc > 0\}$$

which act on vectors $x \in \mathbb{R}^2$ by $x \mapsto Mx$. Two matrices represent the same MT if one is a nonzero multiple of the other and the matrix multiplication corresponds to the composition of MT. If $M \in \mathbb{M}(\mathbb{R})$ and $I \in \mathbb{I}(\mathbb{R})$, then MI is the interval which represents the M-image of the set interval of I.

4 Rational intervals

Denote by $\mathbb{Z}$ the set of integers and by $\overline{\mathbb{Q}} = \{x \in \mathbb{Z}^2 \setminus \{\frac{0}{0}\} :\ \gcd(x) = 1\}$ the set of (homogeneous coordinates of) rational numbers which we understand as a subset of $\overline{\mathbb{R}}$. Here $\gcd(x)$ is the greatest common divisor of x_0 and x_1. The norm of a vector $x \in \overline{\mathbb{Q}}$ is $||x|| = \sqrt{x_0^2 + x_1^2}$. Denote by

$$\begin{aligned}\mathbb{M}(\mathbb{Z}) &= \{M \in \mathrm{GL}(\mathbb{Z},2) :\ \gcd(M) = 1,\ \det(M) > 0\},\\ \mathbb{I}(\mathbb{Z}) &= \{I \in \mathrm{GL}(\mathbb{Z},2) :\ \gcd(I) = 1,\ \det(I) < 0\}.\end{aligned}$$

The norm of a matrix $M_{(a,b,c,d)} \in \mathrm{GL}(\mathbb{Z},2)$ is $||M|| = \sqrt{a^2+b^2+c^2+d^2}$. We have $||MN|| \le ||M||\cdot||N||$ for $M, N \in \mathbb{M}(\mathbb{Z})$.

Lemma 1 *If $I \in \mathbb{I}(\mathbb{Z})$ is an interval, then*

$$\sqrt{2 \cdot |\det(I) \cdot \mathrm{sz}(I)|} \leq ||I|| \leq 2 \cdot |\det(I)| \cdot \max\{|\mathrm{sz}(I)|, 1\}.$$

Proof. Let $I = (\frac{a}{c}, \frac{b}{d})$. Then $2 \cdot |\det(I) \cdot \mathrm{sz}(I)| = 2|ab + cd| \leq ||I||^2$, and we get the first inequality. To prove the second inequality, we show that in all cases $\max\{|a|, |b|, |c|, |d|\} \leq |\det(I)| \cdot \max\{|\mathrm{sz}(I)|, 1\}$. If $a = 0$ or $d = 0$ then $0 \neq |bc| = |\det(I)|$ and $|\det(I) \cdot \mathrm{sz}(I)|$ is either $|cd|$ or $|ab|$ and the claim is satisfied. If $b = 0$ or $c = 0$ then $0 \neq |ad| = |\det(I)|$ and $|\det(I) \cdot \mathrm{sz}(I)|$ is either $|cd|$ or $|ab|$ and the claim is satisfied. If $\mathrm{sgn}(ab) \cdot \mathrm{sgn}(cd) > 0$ then

$$|a| \cdot |b| + |c| \cdot |d| = |ab + cd| = |\mathrm{sz}(I) \cdot \det(I)|,$$

and the claim is satisfied. If $\mathrm{sgn}(ab) \cdot \mathrm{sgn}(cd) < 0$ then $\mathrm{sgn}(ad) \cdot \mathrm{sgn}(bc) = \mathrm{sgn}(abcd) = \mathrm{sgn}(ab) \cdot \mathrm{sgn}(cd) < 0$ and $|a| \cdot |d| + |b| \cdot |c| = |ad - bc| = |\det(I)|$, so the claim is satisfied. □

Lemma 2 *If $I \in \mathbb{I}(\mathbb{Z})$, $\mathrm{sz}(I) < 0$ and $x \in I \cap \overline{\mathbb{Q}}$, then $||I|| \leq \sqrt{5} \cdot ||x|| \cdot |\det(I)|$ and $|\mathrm{sz}(I)| \leq \frac{5}{2}||x||^2 \cdot |\det(I)|$.*

Proof. Let $x = \frac{p}{q} \in I = (\frac{a}{c}, \frac{b}{d})$, and set $\alpha = -\det(\frac{a}{c}, \frac{p}{q}) = pc - aq$, $\beta = -\det(\frac{p}{q}, \frac{b}{d}) = qb - pd$, so $\mathrm{sgn}(\alpha \cdot \beta) > 0$. Replacing x by $\frac{-p}{-q}$ if necessary, we can assume that $\alpha > 0$ and $\beta > 0$. Since $\mathrm{sz}(I) < 0$ and $\mathrm{sz}(\frac{0}{1}, \frac{1}{0}) = 0$, either $0 \notin I$ or $\infty \notin I$. Assume first $\infty \notin I$, so $cd = -\det(\frac{a}{c}, \frac{1}{0}) \cdot \det(\frac{1}{0}, \frac{b}{d}) \geq 0$. Since $q \neq 0$, $a = (pc - \alpha)/q$, $b = (pd + \beta)/q$, and $-\det(I) = (\alpha d + \beta c)/q = (\alpha|d| + \beta|c|)/|q|$, so $\alpha, \beta, |d|, |c|$ are bounded by $|q| \cdot |\det(I)|$. It follows that $|a|$ and $|b|$ are bounded by $(|p| + 1) \cdot |\det(I)|$, so $||I||^2 \leq 2(q^2 + p^2 + 2|p| + 1) \cdot \det(I)^2$. Similarly if $0 \notin I$, then $ab = -\det(\frac{a}{c}, \frac{0}{1}) \cdot \det(\frac{0}{1}, \frac{b}{d}) \geq 0$. Since $p \neq 0$, $c = (aq + \alpha)/p$, $d = (qb - \beta)/p$, and $-\det(I) = (\alpha b + \beta a)/p = (\alpha|b| + \beta|a|)/|p|$, so $\alpha, \beta, |a|, |b|$ are bounded by $|p| \cdot |\det(I)|$. It follows that $|c|$ and $|d|$ are bounded by $(|q| + 1) \cdot |\det(I)|$, so $||I||^2 \leq 2(p^2 + q^2 + 2|q| + 1) \cdot \det(I)^2$. In both cases $||I||^2 \leq 5 \cdot ||x||^2 \cdot \det(I)^2$. Similarly we show that $|\mathrm{sz}(I)| \leq \frac{5}{2}||x||^2 \cdot |\det(I)|$. □

5 Subshifts

For a finite alphabet $\mathbb{A}$ denote by $\mathbb{A}^* := \bigcup_{m \geq 0} \mathbb{A}^m$ the set of finite words. Denote λ the empty word : $\mathbb{A}^0 = \{\lambda\}$. The length of a word $u = u_0 \dots u_{m-1} \in \mathbb{A}^m$ is $|u| = m$. We denote by $\mathbb{A}^{\mathbb{N}}$ the Cantor space of infinite words with the metric $d(u, v) = 2^{-k}$, where $k = \min\{i \geq 0 : u_i \neq v_i\}$. We say that $v \in \mathbb{A}^*$ is a subword of $u \in \mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ and write $v \sqsubseteq u$, if $v = u_{[i,j)} = u_i \dots u_{j-1}$ for some $0 \leq i \leq j \leq |u|$. The cylinder of $u \in \mathbb{A}^n$ is the set $[u] = \{v \in \mathbb{A}^{\mathbb{N}} : v_{[0,n)} = u\}$. The **shift map** $\sigma : \mathbb{A}^{\mathbb{N}} \to \mathbb{A}^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty set $\Sigma \subseteq \mathbb{A}^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. If $D \subseteq \mathbb{A}^*$ then $\Sigma_D = \{x \in \mathbb{A}^{\mathbb{N}} : \forall u \sqsubseteq x, u \notin D\}$ is the subshift (provided it is nonempty) with **forbidden words** D. Any subshift can be obtained in this way. A subshift is uniquely determined by its **language** $\mathcal{L}(\Sigma) = \{u \in \mathbb{A}^* : \exists x \in \Sigma, u \sqsubseteq x\}$. Denote by $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap \mathbb{A}^n$.

A **labelled graph** over an alphabet $\mathbb{A}$ is a structure $\mathcal{G} = (V, E, s, t, \ell)$, where $V = |\mathcal{G}|$ is the set of vertices, E is the set of edges, $s, t : E \to V$ are the source and target maps, and $\ell : E \to \mathbb{A}$ is a labeling function. The subshift of $\mathcal{G}$ consists of all labels of all paths of $\mathcal{G}$. A subshift is **sofic**, if it is the subshift of a finite labelled graph. A subshift Σ is of **finite type** (SFT) of order p, if its forbidden words have length at most p, i.e., if $\Sigma = \Sigma_D$ for some set $D \subset \mathbb{A}^p$. In this case $u \in \mathbb{A}^{\mathbb{N}}$ belongs to Σ iff all subwords of u of length p belong to $\mathcal{L}(\Sigma)$ (see Lind and Marcus [11] or Kůrka [5]).

A **finite state transducer** is a finite state automaton with a read only input tape in an alphabet $\mathbb{A}$ and a write only output tape in an alphabet $\mathbb{B}$. It is given by a finite labelled graph $\mathcal{G}$ with edges $q \xrightarrow{a/b} r$, where $a \in \mathbb{A} \cup \{\lambda\}$ is an input letter and $b \in \mathbb{B} \cup \{\lambda\}$ is an output letter. We say that the transducer is **deterministic** on a subshift $\Sigma \subseteq \mathbb{A}^{\mathbb{N}}$ if for each $q \in V$ and $u \in \Sigma$ there exists a unique $v = F_{\mathcal{G}}(u) \in \mathbb{B}^{\mathbb{N}}$ such that u/v is the label of an infinite path with source q. Such a transducer determines a continuous mapping $F_{\mathcal{G}} : \Sigma \to \mathbb{B}^{\mathbb{N}}$. For any finite state transducer, the computation of $F_{\mathcal{G}}$ has linear time complexity.

6 Möbius number systems

A **Möbius iterative system** over an alphabet $\mathbb{A}$ is a map $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ or a family of orientation-preserving Möbius transformations $(F_u : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{u \in \mathbb{A}^*}$ satisfying $F_{uv} = F_u \circ F_v$ and $F_\lambda = \mathrm{Id}$. An **open almost-cover** is a system of open intervals $\mathcal{W} = \{W_a : a \in \mathbb{A}\}$ indexed by the alphabet $\mathbb{A}$, such that $\bigcup_{a \in \mathbb{A}} \overline{W_a} = \overline{\mathbb{R}}$. If $W_a \cap W_b = \emptyset$ for $a \neq b$, then we say that $\mathcal{W}$ is an **open partition**. We denote by $\mathcal{E}(\mathcal{W}) = \{\mathfrak{l}(W_a), \mathfrak{r}(W_a) : a \in \mathbb{A}\}$ the **set of endpoints** of $\mathcal{W}$.

Definition 1 *A Möbius number system over an alphabet* $\mathbb{A}$ *is a pair* $(F, \mathcal{W})$ *where* $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is a Möbius iterative system and* $\mathcal{W} = \{W_a : a \in \mathbb{A}\}$ *is an almost-cover, such that* $W_a \subseteq \mathbf{V}(F_a)$ *for each* $a \in \mathbb{A}$. *The* **interval cylinder** *of* $u \in \mathbb{A}^{n+1}$ *is* $W_u = W_{u_0} \cap F_{u_0} W_{u_1} \cap \cdots \cap F_{u_{[0,n)}} W_{u_n}$. *The* **expansion subshift** $\mathcal{S}_{\mathcal{W}}$ *is defined by* $\mathcal{S}_{\mathcal{W}} = \{u \in \mathbb{A}^{\mathbb{N}} : \forall k > 0, W_{u_{[0,k)}} \neq \emptyset\}$. *We denote by* $\mathcal{L}_{\mathcal{W}} = \mathcal{L}(\mathcal{S}_{\mathcal{W}})$ *the language of* $\mathcal{S}_{\mathcal{W}}$ *and by* $\mathcal{L}^n_{\mathcal{W}} = \mathcal{L}^n(\mathcal{S}_{\mathcal{W}})$.

For $uv \in \mathcal{L}_{\mathcal{W}}$ we have $W_{uv} = W_u \cap F_u W_v$. Given a MNS $(F, \mathcal{W})$, we construct nondeterministically the expansion $u \in \mathcal{S}_{\mathcal{W}}$ of $x = x_0 \in \overline{\mathbb{R}}$ as follows: Choose u_0 with $x \in W_{u_0}$, choose u_1 with $x_1 = F_{u_0}^{-1}(x_0) \in W_{u_1}$, choose u_2 with $x_2 = F_{u_1}^{-1}(x_1) \in W_{u_2}$, etc. Then $x \in W_{u_{[0,n)}}$ for each n, so W_u is the set of points which have expansion u.

Theorem 2 (Kůrka and Kazda [10]) *If* $(F, \mathcal{W})$ *is a MNS over* $\mathbb{A}$, *then there exists a continuous map* $\Phi : \mathcal{S}_{\mathcal{W}} \to \overline{\mathbb{R}}$ *such that for each* $u \in \mathcal{S}_{\mathcal{W}}$ *and* $v \in \mathcal{L}_{\mathcal{W}}$,

$$\lim_{n \to \infty} F_{u_{[0,n)}}(i) = \Phi(u),\ \{\Phi(u)\} = \bigcap_{n \geq 0} \overline{W_{u_{[0,n)}}},\ \Phi([v] \cap \mathcal{S}_{\mathcal{W}}) = \overline{W_v}.$$

Here i is the imaginary unit. In fact we have $\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(z)$ for each $z \in \mathbb{U}$, and $\mathbf{h}(\Phi(u)) = \lim_{n\to\infty} \widehat{F}_{u_{[0,n)}}(z)$ for each $z \in \mathbb{D}$. If $(F, \mathcal{W})$ is an MNS then $\lim_{n\to\infty} \max\{|W_u| : u \in \mathcal{L}^n_{\mathcal{W}}\} = 0$. This is an immediate consequence of the uniform continuity of $\Phi : \mathcal{S}_{\mathcal{W}} \to \overline{\mathbb{R}}$.

Definition 3 *We say that a MNS $(F, \mathcal{W})$ over $\mathbb{A}$ is an* **integer MNS** *if its transformations have integer entries and its intervals have rational endpoints, i.e., if $F_a \in \mathbb{M}(\mathbb{Z})$ and $W_a \in \mathbb{I}(\mathbb{Z})$ for each $a \in \mathbb{A}$. We say that an integer MNS is* **modular***, if all its transformations have unit determinant* $\det(F_a) = 1$.

7 Sofic Möbius number systems

Definition 4 *Let $(F, \mathcal{W})$ be an MNS over an alphabet $\mathbb{A}$. An open partition $\mathcal{V} = \{V_p : p \in \mathbb{B}\}$ is an* **SFT refinement** *of $\mathcal{W}$, if the following two conditions are satisfied for each $a \in \mathbb{A}$, $p, q \in \mathbb{B}$:*
1. If $V_p \cap W_a \neq \emptyset$ then $V_p \subseteq W_a$,
2. If $V_p \subseteq W_a$ and $V_q \cap F_a^{-1} V_p \neq \emptyset$ then $V_q \subseteq F_a^{-1} V_p$.
In this case we say that $(F, \mathcal{W}, \mathcal{V})$ is a **sofic Möbius number system**. *The* **base** *graph $\mathcal{G}_{(\mathcal{W},\mathcal{V})}$ of $(F, \mathcal{W}, \mathcal{V})$ is an $\mathbb{A}$-labelled graph whose set of vertices are letters of $\mathbb{B}$ and whose labelled edges are $p \overset{a}{\to} q$ if $F_a V_q \subseteq V_p \subseteq W_a$. Denote by $\mathbb{C} = \{(p, a) \in \mathbb{B} \times \mathbb{A} : V_p \subseteq W_a\}$ and $\mathcal{S}_{(\mathcal{W},\mathcal{V})} \subseteq \mathbb{C}^{\mathbb{N}}$ the SFT of order two with transitions $(p, a) \to (q, b)$ iff $p \overset{a}{\to} q$.*

Theorem 5 (Kůrka [9]) *If $(F, \mathcal{W})$ is an MNS, then $\mathcal{S}_{\mathcal{W}}$ is a sofic subshift iff there exists an SFT refinement $\mathcal{V}$ of $\mathcal{W}$. In this case $\mathcal{S}_{\mathcal{W}}$ is the subshift of the base graph $\mathcal{G}_{(\mathcal{W},\mathcal{V})}$ and we have a factor map $\pi : \mathcal{S}_{(\mathcal{W},\mathcal{V})} \to \mathcal{S}_{\mathcal{W}}$ given by $\pi(p, a) = a$.*

Theorem 6 (Kůrka [9], Theorem 16) *Each modular MNS has a sofic expansion subshift.*

An example of a modular MNS has been studied by Raney [14], Niqui [12] and Kůrka [9]. Its alphabet is $\mathbb{A} = \{0, 1, 2, 3\}$, the transformations are

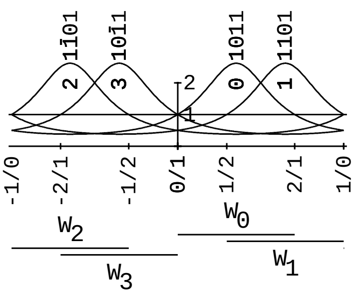
$$F_0(x) = \frac{x}{1+x},\ F_1(x) = x+1,\ F_2(x) = x-1,\ F_3(x) = \frac{x}{1-x}.$$

and the intervals are $W_0 = (0, 1)$, $W_1 = (1, \infty)$, $W_2 = (\infty, -1)$, $W_3 = (-1, 0)$. Since $F_a(0, \infty) = W_a$ for $a = 0, 1$ and $F_a(\infty, 0) = W_a$ for $a = 2, 3$, the expansion subshift is a union of two full subshifts which code respectively nonnegative and nonpositive real numbers: $\mathcal{S}_{\mathcal{W}} = \{0, 1\}^{\mathbb{N}} \cup \{2, 3\}^{\mathbb{N}}$. The system is closely related to continued fractions. Each $u \in \{0, 1\}^{\mathbb{N}}$ can be written as $u = 1^{a_0} 0^{a_1} 1^{a_2} \ldots$, where $a_0 \geq 0$ and $a_n > 0$ for $n > 0$. Then u is the expansion of the continued fraction $[a_0, a_1, a_2, \ldots]$, i.e.,

$$\Phi(u) = [a_0, a_1, a_2, \ldots] = a_0 + 1/(a_1 + 1/(a_2 + \cdots.$$

If $a_n = \infty$ for some $n > 0$, then $\Phi(u) = [a_0, \ldots, a_{n-1}]$ is a finite continued fraction.

a	F_a	W_a	$F_a^{-1}W_a$
0	$[1,0,1,1]$	$(\frac{0}{1},\frac{2}{1})$	$(\frac{0}{1},\frac{2}{-1})$
1	$[1,1,0,1]$	$(\frac{1}{2},\frac{1}{0})$	$(\frac{-1}{2},\frac{1}{0})$
2	$[1,-1,0,1]$	$(\frac{-1}{0},\frac{-1}{2})$	$(\frac{-1}{0},\frac{1}{2})$
3	$[1,0,-1,1]$	$(\frac{-2}{1},\frac{0}{1})$	$(\frac{2}{1},\frac{0}{-1})$



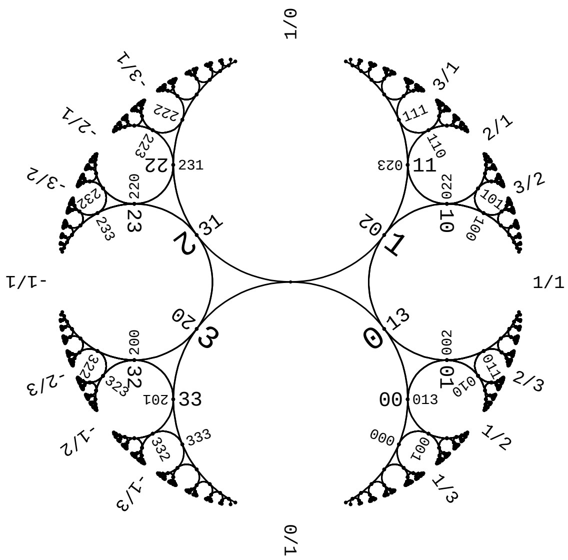


Fig. 1. A modular MNS.

In Figure 1 we show a variant of this system with larger cylinder intervals $W_a = \mathbf{V}(F_a)$. Figure 1 bottom left shows the graphs of the circle derivations $(F_a^{-1})^\bullet(x)$ together with the cylinder intervals W_a. In Figure 1 right we can see the values $\widehat{F}_u(0)$ of the disc MT $\widehat{F}_u$ at zero. The curves between $\widehat{F}_u(0)$ are constructed as follows. For each MT M there exists a family $(M^r)_{r\in\mathbb{R}}$ of MT such that $M^0 = \mathrm{Id}$, $M^1 = M$, and $M^{r+s} = M^r M^s$. Each value $\widehat{F}_u(0)$ is joined to $\widehat{F}_{ua}(0)$ by the curve $(\widehat{F}_u\widehat{F}_a^r(0))_{0\le r\le 1}$. The labels $u \in \mathbb{A}^*$ at $\widehat{F}_u(0)$ are written in the direction of the tangent vectors $\widehat{F}'_u(0)$. The SFT partition of the system has 8 intervals shown in Figure 2 left. The base graph can be seen in Figure 2 right. The expansion subshift $\mathcal{S}_\mathcal{W}$ is a SFT of order 4. with 20 forbidden words 03, 12, 21, 30, 020, 131, 202, 313, 0220, 0232, 0233, 1322, 1323, 1331, 2002, 2010, 2011, 3100, 3101, 3113.

Theorem 7 *If $(F,\mathcal{W},\mathcal{V})$ is a modular system, then $\pi : \mathcal{S}_{(\mathcal{W},\mathcal{V})} \to \mathcal{S}_\mathcal{W}$ is an isomorphism, so $\mathcal{S}_\mathcal{W}$ is an SFT.*

Proof. We show that if $(p,u) \in \mathcal{S}_{(\mathcal{W},\mathcal{V})}$, then $p \in \mathbb{B}^\mathbb{N}$ is determined by $u \in \mathbb{A}^\mathbb{N}$. For $0 \le n < m$ we have $V_{p_n} \subseteq W_{u_n}$ and $F_{u_n}V_{p_{n+1}} \subseteq V_{p_n}$, so

$$F_{u_{[n,m)}}V_{p_m} \subseteq F_{u_{[n,m-1)}}V_{p_{m-1}} \subseteq \cdots \subseteq F_{u_n}V_{p_{n+1}} \subseteq V_{p_n},$$
$$F_{u_{[n,m)}}V_{p_m} \subseteq F_{u_{[n,m-1)}}W_{u_{m-1}} \cap \cdots \cap F_{u_n}W_{u_{n+1}} \cap W_{u_n} \subseteq W_{u_{[n,m)}}.$$

It follows that $\emptyset \ne F_{u_{[n,m)}}V_{p_m} \subseteq V_{p_n} \cap W_{u_{[n,m)}}$ is nonempty. Denote by $x_n = \Phi(\sigma^n(u))$, so $\{x_n\} = \bigcap_{m>n}\overline{W_{u_{[n,m)}}}$. If x_n is irrational, then there exists $m > n$ such that $W_{u_{[n,m)}} \cap \mathcal{E}(\mathcal{V}) = \emptyset$, so there exists exactly one $p_n \in \mathbb{B}$ with $V_{p_n} \cap$

pa	V_p	F_a	$F_a^{-1}V_p$	followers
00	$(\frac{0}{1}, \frac{1}{2})$	$[1,0,1,1]$	$(\frac{0}{1}, \frac{1}{1})$	0, 1
10	$(\frac{1}{2}, \frac{1}{1})$	$[1,0,1,1]$	$(\frac{1}{1}, \frac{1}{0})$	2, 3
11	$(\frac{1}{2}, \frac{1}{1})$	$[1,1,0,1]$	$(\frac{-1}{2}, \frac{0}{1})$	7
20	$(\frac{1}{1}, \frac{2}{1})$	$[1,0,1,1]$	$(\frac{1}{0}, \frac{2}{-1})$	4
21	$(\frac{1}{1}, \frac{2}{1})$	$[1,1,0,1]$	$(\frac{0}{1}, \frac{1}{1})$	0, 1
31	$(\frac{2}{1}, \frac{1}{0})$	$[1,1,0,1]$	$(\frac{1}{1}, \frac{1}{0})$	2, 3
42	$(\frac{-1}{0}, \frac{-2}{1})$	$[1,-1,0,1]$	$(\frac{-1}{0}, \frac{-1}{1})$	4, 5
52	$(\frac{-2}{1}, \frac{-1}{1})$	$[1,-1,0,1]$	$(\frac{-1}{1}, \frac{0}{1})$	6, 7
53	$(\frac{-2}{1}, \frac{-1}{1})$	$[1,0,-1,1]$	$(\frac{-2}{-1}, \frac{-1}{0})$	3
62	$(\frac{-1}{1}, \frac{-1}{2})$	$[1,-1,0,1]$	$(\frac{0}{1}, \frac{1}{2})$	0
63	$(\frac{-1}{1}, \frac{-1}{2})$	$[1,0,-1,1]$	$(\frac{-1}{0}, \frac{-1}{1})$	4, 5
73	$(\frac{-1}{2}, \frac{0}{1})$	$[1,0,-1,1]$	$(\frac{-1}{1}, \frac{0}{1})$	6, 7

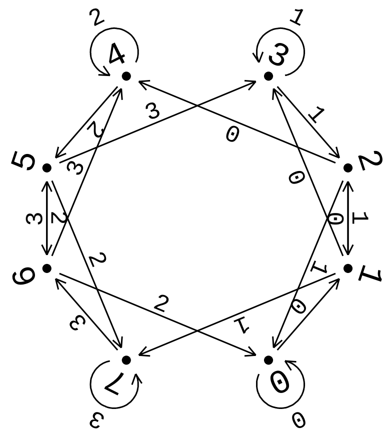


Fig. 2. The SFT partition and the base graph of a modular system from Figure 1.

$W_{u_{[n,m)}} \neq \emptyset$. Assume that x_n is rational. For each $m > n$ we have

$$x_m = \Phi(\sigma^m(u)) = F^{-1}_{u_{[n,m)}}(x_n) \in \overline{W_{u_m}} \subseteq \overline{\mathbf{V}(F_{u_m})},$$

and $||x_m||^2/||x_{m+1}||^2 = ||x_m||^2/||F^{-1}_{u_m}(x_m)||^2 = (F^{-1}_{u_m})^{\bullet}(x_m) \geq 1$, so $||x_{m+1}|| \leq ||x_m||$. Moreover, if $x_m \in W_{u_m}$, then $||x_{m+1}|| < ||x_m||$. Since $||x_m||^2 \in \mathbb{N}$, the set $\{m \geq n :\ x_m \in W_{u_m}\}$ is finite and there exists $m > n$ such that either $x_k = \mathfrak{l}(W_{u_k})$ for all $k \geq m$, or $x_k = \mathbf{r}(W_{u_k})$ for all $k \geq m$. Since $x_n = F_{u_{[n,k)}}(x_k) \in \overline{W_{u_{[n,k)}}} \subseteq F_{u_{[n,k)}}\overline{W_{u_k}}$, we get $x_n = \mathfrak{l}(W_{u_{[n,k)}})$ for all $k \geq m$ in the former case and $x_n = \mathbf{r}(W_{u_{[n,k)}})$ for all $k \geq m$ in the latter case. It follows that there exists $k > m$ such that $W_{u_{[n,k)}} \cap \mathcal{E}(\mathcal{V}) = \emptyset$, so there exists a unique p_n with $V_{p_n} \cap W_{u_{[n,k)}} \neq \emptyset$. This means that p_n is uniquely determined by u. Since $F_{u_{n-1}}V_{p_n} \subseteq V_{p_{n-1}}$, the letter p_{n-1} is uniquely determined by p_n and the prefix $p_{[0,n)}$ of p is uniquely determined by p_n. □

Theorem 8 *Assume that $(F, \mathcal{W}, \mathcal{V})$ is a modular MNS and for $u \in \mathcal{L}_{\mathcal{W}}$ denote by $\mathcal{P}(u) \subseteq \mathbb{B}^*$ the set of paths with label u.*
1. There exists $r > 0$ such that the set $\{p_{[0,n-r]} :\ p \in \mathcal{P}(u)\}$ is a singleton for each $n > r$ and each finite word $u \in \mathcal{L}^n_{\mathcal{W}}$.
2. There exists $s > 0$ such that $\mathcal{P}(u)$ has at most s elements for each $u \in \mathcal{L}_{\mathcal{W}}$.
3. The map $\pi^{-1} : \mathcal{S}_{\mathcal{W}} \to \mathcal{S}_{(\mathcal{W},\mathcal{V})}$ can be computed by a finite state transducer.

Proof. The existence of constants r, s follows from Theorem 7 by a compactness argument. We define a finite state transducer for π^{-1} as follows. Its vertices are sets $X \subseteq \mathbb{B}^n$, where $0 < n \leq r$. The labelled edges are

$$X \xrightarrow{a/\lambda} \{p \in \mathbb{B}^{n+1} :\ p_{[0,n-1]} \in X,\ p_{n-1} \overset{a}{\to} p_n\} \quad \text{if } X \subseteq \mathbb{B}^n,\ n < r,$$

$$X \xrightarrow{a/b} \{p \in \mathbb{B}^{r} :\ bp_{[0,r-2]} \in X,\ p_{r-2} \overset{a}{\to} p_{r-1}\} \quad \text{if } X \subseteq \mathbb{B}^r.$$

Then u/p is the label of a path with the source $\mathbb{B}$ iff p is a prefix of a path whose label is u. □

In Table 2 left we show the computation of $\pi^{-1}(u)$ on input word $u = 00133$. For each $n > 0$ we give the set $\mathcal{P}(u_{[0,n)})$ of all paths $p \in \mathbb{B}^{n+1}$ with label $u_{[0,n)}$.

8 Arithmetical algorithms

Definition 9 *The* **unary graph** *for an integer sofic MNS $(F, \mathcal{W}, \mathcal{V})$ is a labelled graph whose vertices are (X, p), where $X \in \mathbb{M}(\mathbb{Z})$ and $p \in \mathbb{B}$. Its labelled edges are*

$$\begin{aligned} \textit{absorption: } & (X,p) \xrightarrow{a/\lambda} (XF_a, q) \quad \textit{if } F_aV_q \subseteq V_p \subseteq W_a, \\ \textit{emission: } & (X,p) \xrightarrow{\lambda/b} (F_b^{-1}X, p) \quad \textit{if } XV_p \subseteq W_b. \end{aligned}$$

The labels of paths are concatenations of the labels of their edges. They have the form u/v where $u \in \mathcal{L}_\mathcal{W}$ is an input word and $v \in \mathcal{L}_\mathcal{W}$ is an output word.

Proposition 10 *If $(X,p) \xrightarrow{u/v} (Y,q)$ is a path in the unary graph, then*

$$Y = F_v^{-1}XF_u,\ F_uV_q \subseteq V_p \cap W_u,\ XF_uV_q \subseteq W_v.$$

Proof. Since $W_\lambda = \overline{\mathbb{R}}$ and $F_\lambda = \mathrm{Id}$, the statement holds for the absorption and emission edges. Assume by induction that the statement holds for a path with label u/v. If $(X,p) \xrightarrow{u/v} (Y,q) \xrightarrow{a/\lambda} (Z,r)$ then $Z = YF_a = F_v^{-1}XF_{ua}$, $F_aV_r \subseteq V_q \subseteq W_a$, so $F_{ua}V_r \subseteq F_uV_q \subseteq V_p \cap W_u \cap F_uW_a = V_p \cap W_{ua}$, and $XF_{ua}V_r \subseteq XF_uV_q \subseteq W_v$, so the statement holds for $(X,p) \xrightarrow{ua/v} (Z,r)$. If $(X,p) \xrightarrow{u/v} (Y,q) \xrightarrow{\lambda/b} (Z,q)$ then $Z = F_b^{-1}Y = F_{vb}^{-1}XF_u$. From $F_v^{-1}XF_uV_q = YV_q \subseteq W_b$ we get $XF_uV_q \subseteq F_vW_b$, and therefore $XF_uV_q \subseteq W_v \cap F_vW_b = W_{vb}$. Moreover, $F_uV_q \subseteq V_p \cap W_u$, so the statement holds for $(X,p) \xrightarrow{u/vb} (Z,q)$. □

```
procedure unary;
input: M ∈ 𝕄(ℤ), (p,u) ∈ 𝒮_(𝒲,𝒱) ∪ ℒ_(𝒲,𝒱);
output: v ∈ 𝒮_𝒲 ∪ ℒ_𝒲;
variables X ∈ 𝕄(ℤ) (state), n, m ∈ ℕ (input and output pointers);
begin
  X := M; n := 0; m := 0;
  while n < |u| repeat
    if ∀a ∈ 𝔸, XV_{p_n} ⊈ W_a then begin
      X := XF_{u_n}; n := n + 1; end;
    else begin
      v_m := a, where XV_{p_n} ⊆ W_a and XV_{p_n} ⊈ W_b for all b < a;
      X := F_a^{-1}X; m := m + 1; end;
    end;
```

Table 1. The unary algorithm.

n	$\mathcal{P}(001333_{[0,n)})$
1	$00, 01, 12, 13, 24,$
2	$000, 001, 012, 013, 124,$
3	$0017, 0120, 0121,$
	$0132, 0133,$
4	$00176, 00177,$
5	$001764, 001765,$
	$001776, 001777,$
6	$0017653, 0017764,$
	$0017765, 0017776,$
	$0017777,$

n	m	out	X	XV_{p_n}	input
0	0	0	$[2, 1, 1, 2]$	$(\frac{1}{2}, \frac{4}{5})$	0
0	1	1	$[2, 1, -1, 1]$	$(\frac{1}{1}, \frac{4}{1})$	0
0	2		$[3, 0, -1, 1]$	$(\frac{0}{1}, \frac{3}{1})$	$0 \xrightarrow{0} 0$
1	2	0	$[3, 0, 0, 1]$	$(\frac{0}{1}, \frac{3}{2})$	0
1	3		$[3, 0, -3, 1]$	$(\frac{0}{1}, \frac{3}{-1})$	$0 \xrightarrow{0} 1$
2	3	2	$[3, 0, -2, 1]$	$(\frac{3}{0}, \frac{3}{-1})$	1
2	4	2	$[1, 1, -2, 1]$	$(\frac{3}{0}, \frac{2}{-1})$	1
2	5	2	$[-1, 2, -2, 1]$	$(\frac{3}{0}, \frac{1}{-1})$	1
2	6		$[-3, 3, -2, 1]$	$(\frac{3}{0}, \frac{0}{-1})$	$1 \xrightarrow{1} 7$
3	6		$[-3, 0, -2, -1]$	$(\frac{3}{0}, \frac{0}{-1})$	$7 \xrightarrow{3} 7$

Table 2. The computation of a path $p = 0017 = \pi^{-1}(001333) = \pi^{-1}(u)$ (left) and the computation of $v = 010222 = \Theta_M(p, u)$ on the input matrix $M(x) = (2x+1)/(x+2)$ and the input path $0 \xrightarrow{0} 0 \xrightarrow{0} 1 \xrightarrow{1} 7 \xrightarrow{3} 7$ by the unary algorithm (right). The third column gives the values v_m on emission steps and the empty word on absorption steps. The last column gives the vertex p_n on emission steps and the edge $p_n \xrightarrow{u_n} p_{n+1}$ on absorption steps.

We consider a deterministic **unary algorithm** given in Table 1, which computes a path in the unary graph. Its input is a matrix $M \in \mathbb{M}(\mathbb{Z})$ and either a finite path $(p, u) \in \mathcal{L}_{(\mathcal{W},\mathcal{V})}$ or an infinite path $(p, u) \in \mathcal{S}_{(\mathcal{W},\mathcal{V})}$. We assume that the alphabet $\mathbb{A}$ is linearly ordered. At each step, the algorithm performs the first possible emission if there is one, and an absorption if there is no emission applicable. For an infinite input path, the algorithm computes an output word $v \in \mathcal{S}_{\mathcal{W}}$ such that u/v is the label of a path in the unary graph with source (M, p_0). An example of the computation of the unary algorithm is given in Table 2 right. We are going to prove that for a modular system $(F, \mathcal{W}, \mathcal{V})$, the norm of the state matrix X remains bounded during the computation of the unary algorithm. To do so, we define some constants and prove several lemmas. Set

$$\begin{aligned} B_0 &= \max\{\sqrt{5} \cdot ||x|| : \ x \in \mathcal{E}(\mathcal{W})\}, & B_1 &= \max\{1, |\mathrm{sz}(F_b^{-1} W_b)| : \ b \in \mathbb{A}\} \\ D_0 &= \min\{|\det(V_p)| : \ p \in \mathbb{B}\}, & D_1 &= \max\{|\det(V_p)| : \ p \in \mathbb{B}\}, \\ G &= \max\{1, ||V_p^{-1} F_a V_q|| : \ p \xrightarrow{a} q\}, & H &= \max\{\sqrt{D_0}, ||V_p|| : \ p \in \mathbb{B}\}, \\ B &= \max\{B_0, 2B_1\}, & C_0 &= \max\{B^2 D_1^2 G^2 / 2D_0, B_1\} \end{aligned}$$

Lemma 3 *1. If* $(X, p) \xrightarrow{a/\lambda} (XF_a, q)$, *then* $\mathrm{sz}(XF_aV_q) < \mathrm{sz}(XV_p)$.
2. If $(X, p) \xrightarrow{\lambda/b} (F_b^{-1}X, p)$, *then* $0 > \mathrm{sz}(XV_p) < \mathrm{sz}(F_b^{-1}XV_p) < B_1$.

Proof. The first claim follows from $XF_aV_q \subseteq XV_p$. To prove the second claim, note that for each $M \in \mathbb{M}(\mathbb{Z})$ we have $\mathrm{sz}(\mathbf{V}(M)) < 0$, so $\mathrm{sz}(W_b) < 0$ for each $b \in \mathbb{A}$. If $(X, p) \xrightarrow{\lambda/b} (F_b^{-1}X, p)$ is an emission edge, then $XV_p \subseteq W_b$, so $\mathrm{sz}(XV_p) < 0$. Since $F_b^{-1}XV_p \subseteq F_b^{-1}W_b$, we get $\mathrm{sz}(F_b^{-1}XV_p) < B_1$. Since F_b^{-1} is an expansion on W_b, we get $\mathrm{sz}(XV_p) < \mathrm{sz}(F_b^{-1}XV_p)$. □

Lemma 4 *If* $(X,p) \xrightarrow{a/\lambda} (XF_a, q)$ *is an absorption performed by the unary algorithm and* $\mathrm{sz}(XV_p) < B_1$*, then* $||XV_p|| < BD_1 \det(X)$*,* $|\mathrm{sz}(XV_p)| < C_0 \det(X)$ *and* $|\mathrm{sz}(XF_aV_q)| < C_0 \det(X)$*.*

Proof. We distinguish two cases. If $0 \leq \mathrm{sz}(XV_p) < B_1$, then by Lemma 1 we have $||XV_p|| < 2|\det(XV_p)| \cdot \max\{1, |\mathrm{sz}(XV_p)|\} \leq 2B_1D_1 \det(X)$. If $\mathrm{sz}(XV_p) < 0$, then we use the fact that XV_p is not contained in any W_a, so it must contain a point from $\mathcal{E}(\mathcal{W})$. By Lemma 2, $||XV_p|| \leq B_0 \cdot |\det(XV_p)| \leq B_0D_1 \det(X)$. Thus in both cases we have $||XV_p|| \leq BD_1 \det(X)$. It follows $||XF_aV_q|| \leq ||XV_p|| \cdot ||V_p^{-1}F_aV_q|| \leq BD_1G \cdot \det(X)$. By Lemma 1 we get $|\mathrm{sz}(XV_p)| \leq ||XV_p||^2/2|\det(XV_p)| \leq \frac{B^2D_1^2}{2D_0} \det(X) \leq C_0 \det(X)$, and similarly $|\mathrm{sz}(XF_aV_q)| \leq \frac{B^2D_1^2G^2}{2D_0} \det(X) \leq C_0 \det(X)$. □

Lemma 5 *Every infinite path computed by the unary algorithm contains an infinite number of emissions.*

Proof. Assume by contradiction that there exists an infinite path of absorptions with vertices (X_n, p_n) and label u/λ, where $u \in \mathcal{S}_\mathcal{W}$. Since $F_{u_{[0,n)}}V_{p_n} \subseteq W_{u_{[0,n)}}$ and $\lim_{n\to\infty} |W_{u_{[0,n)}}| = 0$, we get $\lim_{n\to\infty} |X_0F_{u_{[0,n)}}V_{p_n}| = 0$ by the continuity of X_0, and therefore $\lim_{n\to\infty} \mathrm{sz}(X_0F_{u_{[0,n)}}V_{p_n}) = -\infty$. This is in a contradiction with Lemma 4. □

Theorem 11 *For a modular MNS* $(F, \mathcal{W}, \mathcal{V})$ *there exists a constant* $C > 0$ *such that for every input matrix* $M \in \mathbb{M}(\mathbb{Z})$*, the unary algorithm computes a continuous function* $\Theta_M : \mathcal{S}_{(\mathcal{W},\mathcal{V})} \to \mathcal{S}_\mathcal{W}$ *with* $\Phi\Theta_M(p,u) = M\Phi(u)$*, and the state matrix* X *satisfies* $||X|| < C \cdot \max\{||M||^2, \det(M)^2\}$ *during the computation.*

Proof. Let (X_n, p_n) be the vertices of the infinite path with source $(X_0, p_0) = (M, p_0)$. If $\mathrm{sz}(X_nV_{p_n}) > C_0 \det(M)$, then (X_n, p_n) is an absorption vertex by Lemma 3 and $\mathrm{sz}(X_{n+1}V_{p_{n+1}}) < \mathrm{sz}(X_nV_{p_n})$. If $\mathrm{sz}(X_nV_{p_n}) < -C_0 \det(M)$, then (X_n, p_n) is an emission vertex by Lemma 4, and $\mathrm{sz}(X_{n+1}V_{p_{n+1}}) > \mathrm{sz}(X_nV_{p_n})$. Thus there exists m, such that for all $n \geq m$ we have $|\mathrm{sz}(X_nV_{p_n})| < C_0 \det(M)$ while for $n < m$ we have $|\mathrm{sz}(X_nV_{p_n})| \leq |\mathrm{sz}(MV_{p_0})| \leq \frac{H^2 \cdot ||M||^2}{2D_0 \det(M)}$. By Lemma 1 we get either $||X_nV_{p_n}|| \leq 2D_1C_0 \det(M)^2$ in the former case and $||X_nV_{p_n}|| \leq \frac{H^2D_1}{D_0}||M||^2$ in the latter case. Taking $C = \max\{2HD_1C_0, H^3D_1/D_0\}$ we get

$$||X_n|| \leq ||X_nV_{p_n}|| \cdot ||V_{p_n}^{-1}|| \leq C \cdot \max\{||M||^2, \det(M)^2\}$$

for all n, so the algorithm can be realized by a finite state transducer. By Lemma 5, for each $(p,u) \in \mathcal{S}_{(\mathcal{W},\mathcal{V})}$ there exists a unique $v = \Theta_M(p,u)$ such that u/v is the label of an infinite path with source (M, p_0). For each m there exists n such that $u_{[0,n)}/v_{[0,m)}$ is the label of a finite path with source (M, p_0), $\emptyset \neq F_{u_{[0,n)}}V_{p_n} \subseteq W_{u_{[0,n)}}$, and $\emptyset \neq MF_{u_{[0,n)}}V_{p_n} \subseteq W_{v_{[0,m)}}$. The intersection $\bigcap_n F_{u_{[0,n)}}\overline{V_{p_n}} \subseteq \bigcap_n \overline{W_{u_{[0,n)}}}$ is nonempty by compactness and has zero diameter, so it contains the unique point $\Phi(u)$. The intersection $\bigcap_n MF_{u_{[0,n)}}\overline{V_{p_n}} \subseteq \bigcap_m \overline{W_{v_{[0,m)}}}$ is a nonempty singleton which contains both $M(\Phi(u))$ and $\Phi(v)$, so $M(\Phi(u)) = \Phi(v)$. □

Corollary 12 *If $(F, \mathcal{W}, \mathcal{V})$ is a modular MNS, then for each $M \in \mathbb{M}(\mathbb{Z})$ there exists a finite state transducer which computes a continuous function $\Psi_M : \mathcal{S}_\mathcal{W} \to \mathcal{S}_\mathcal{W}$ which satisfies $\Phi\Psi_M = M\Phi$.*

Proof. Using Theorems 8 and 11 we get $\Psi_M = \Theta_M \circ \pi^{-1}$.

A disadvantage of modular systems is that they are not redundant. As shown in Kůrka [7], the cylinder intervals of a modular system contain neither 0 nor ∞, so they cannot form a cover but only an almost-cover. In Kůrka [8] we argue that in some redundant MNS, the unary algorithm has asymtotically linear time complexity. The norm of the state matrix remains small most of the time, although fluctuations to larger values occur sporadically.

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