

On graphs coverable with k shortest paths

Maël Dumas¹, Florent Foucaud², Anthony Perez¹, Ioan Todinca¹

¹LIFO, Université d'Orléans, France

²LIMOS, Université Clermont Auvergne, France

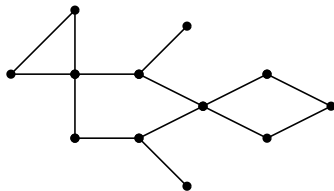
- 1 Graphs coverable by k shortest paths
 - Graphs edge-coverable by k shortest paths
 - Graphs vertex-coverable by k shortest paths

- 2 Algorithmic Consequences

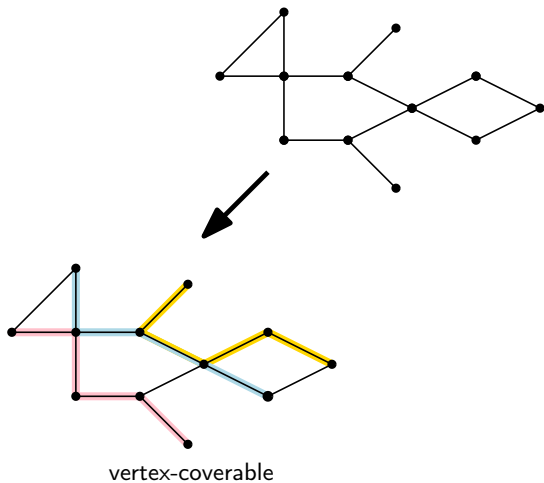
- 3 Conclusion

Graphs coverable by k shortest paths

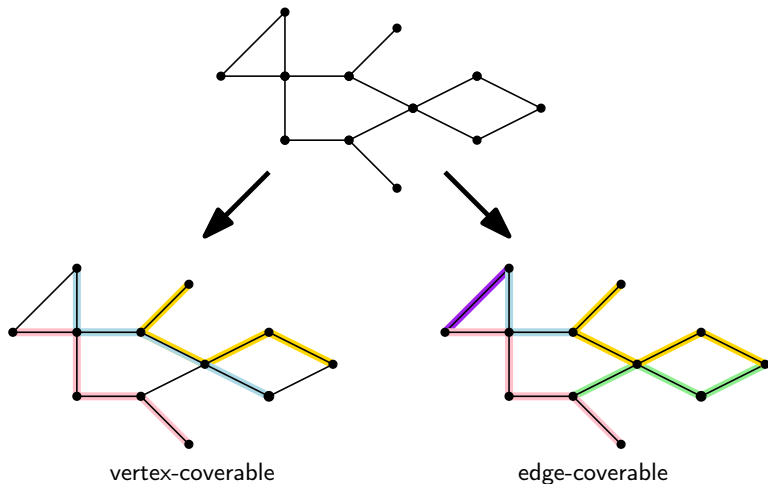
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Combinatorial results

Theorem 1

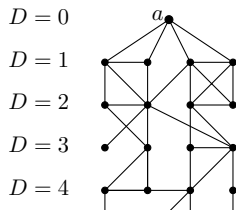
If G is **coverable by k shortest paths** then, for any vertex a and fixed distance D , the number of vertices at distance exactly D from a is upper bounded by some function $g(k)$.

Edge-coverable : $g(k) = O(3^k)$.

Vertex-coverable : $g(k) = O(k \cdot 3^k)$.

Corollary 1

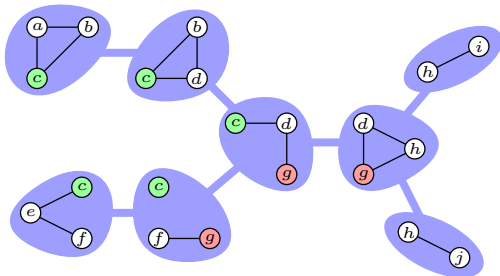
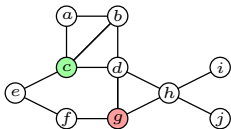
G is of **treewidth** at most $2 \cdot g(k) - 1$.



Treewidth

Tree decomposition :

- Each vertex and each edge is in at least on bag
- The bags containing a vertex v induce a connected subtree



Width of a decomposition : size of the biggest bag - 1

Treewidth : smallest width among all tree decomposition

Combinatorial results

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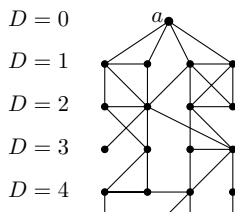
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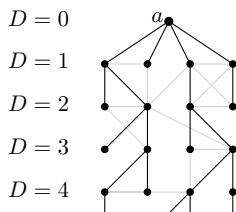
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Tree decomposition :

Do a a breadth-first search (BFS)
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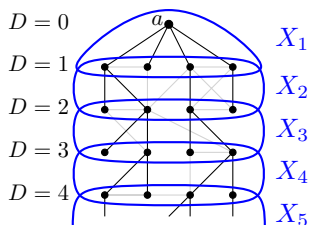
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Tree decomposition :

Do a a breadth-first search (BFS)
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Each bag : two consecutive layers.

Graphs edge-coverable by k shortest paths

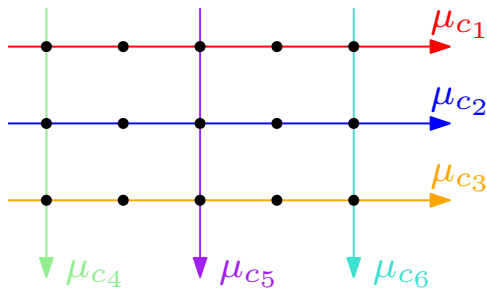
Base paths colouring

Base paths : k shortest paths μ_1, \dots, μ_k that cover the graph

To each base path μ_c we give :

A **colour** c , $1 \leq c \leq k$,

An arbitrary **direction**.

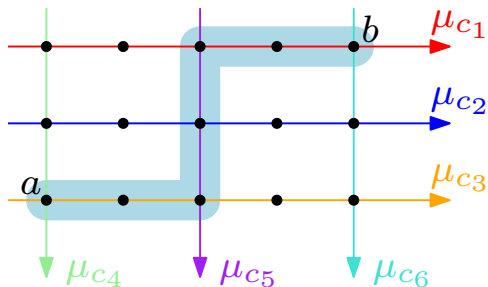


For an edge e of the graph : $\text{colours}(e) = \{i \mid \mu_i \text{ contains } e\}$.

Colouring of a path

Colouring col of P : $\forall e \in E(P), \text{col}(e) \in \text{colours}(e)$

Coloured path : $(P; \text{col})$

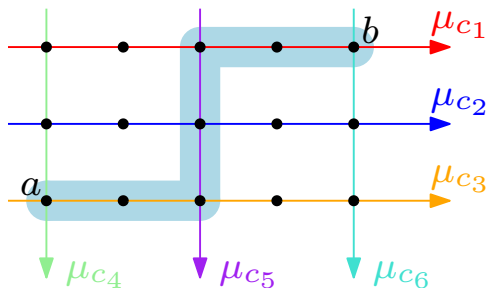


Colours-signs word

For a coloured path $(P; \text{col})$:

divide it in monochromatic subpath of colour c_i ,

each subpath induce a "+" sign or a "-" sign w.r.t. the direction of c_i .



$$\omega = ((c_3, +), (c_5, -), (c_1, +))$$

Good colouring of a path

Well-coloured : the edges using a colour c form a **connected subpath** of P .

A path well-coloured :



A path not well-coloured :



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Good colouring Lemma

For every pair of vertices $a; b$, there exists a **shortest well-coloured** a - b path.

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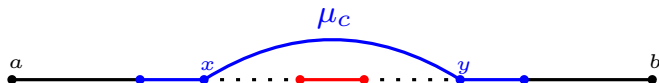


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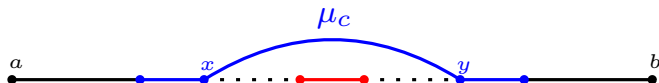
) Replace $P[x; y]$ by ${}_c[x; y]$.

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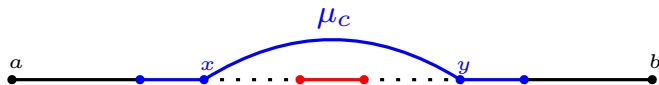
The constructed path is still a shortest path ($|c[x; y]| \leq |P[x; y]|$).

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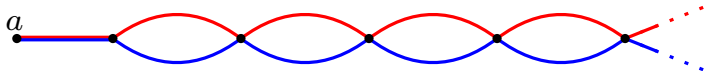
) Replace $P[x; y]$ by $\mu_c[x; y]$.

The constructed path is still a shortest path ($\mu_c[x; y]$ is $\leq P[x; y]$).

The number of colours-signs words possible on every well-coloured paths is upper bounded by $\sum_{l=1}^k 2^l \frac{k!}{(k-l)!} = O(k^k)$.

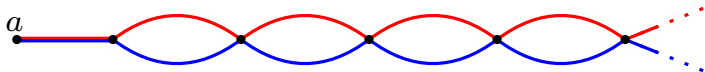
A first bound

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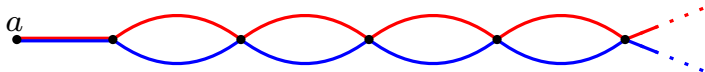


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The shortest paths starting at a vertex a , of length D and colours-signs word $!$ all ends at the same vertex b .

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(weak) Theorem 1

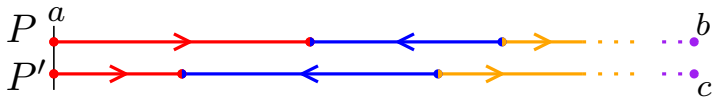
For any vertex a and any fixed distance D , the number of vertices at distance exactly D from a is upper bounded by $O(k^k)$ (number of colours-signs words).

Proof of the Colours-signs word Lemma

Let b and c be vertices at same distance from a vertex a of G .

Let $(P; \text{col})$, $(P^0; \text{col}^0)$ be a well-coloured shortest a - b and a - c paths.

Claim : If they have the same colours-signs word, then $b = c$.



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Proof by induction on the number of letters ℓ of the colours-signs word.

If $\ell = 1$



$\text{dist}(a; b) = \text{dist}(a; c)$ thus $b = c$.

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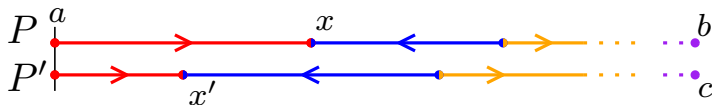
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Take P the path with the longest subpath of the colour c_1 .

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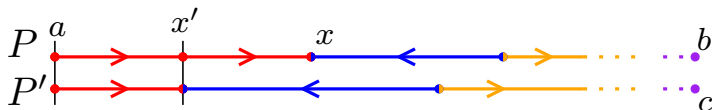
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The vertex x^0 is in the path $c_1[a; x]$, thus in the path P

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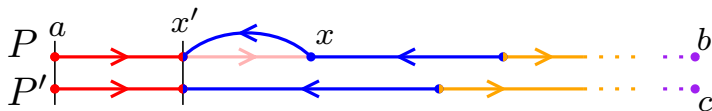
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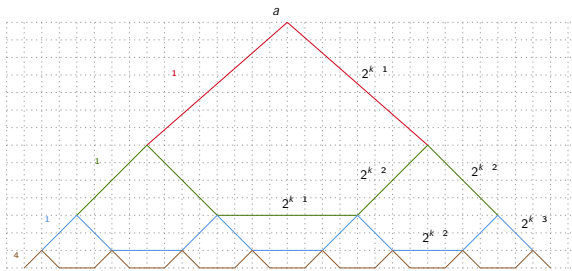
Replace $P[x^0; x]$ by $c_2[x^0; x]$.

$P[x^0; b]$ and $P^0[x^0; c]$ have $\ell - 1$ colours, by the induction hypothesis, $b = c$.

A better bound ?

We shown an the upper bound : $O(k^k)$

Lower bound : $O(2^k)$

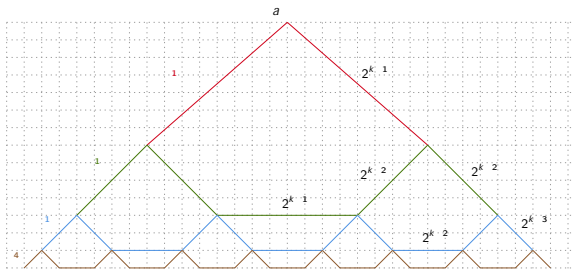


Goal : Single exponential bound.

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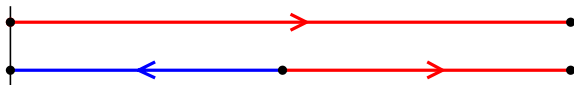
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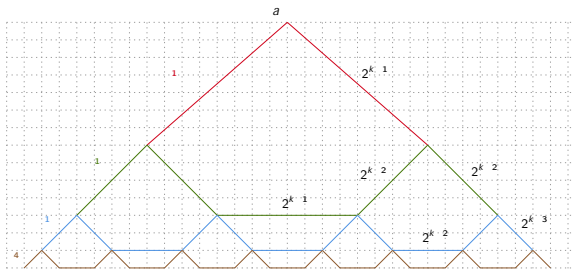
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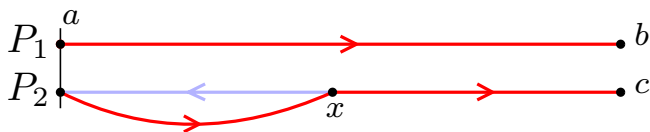
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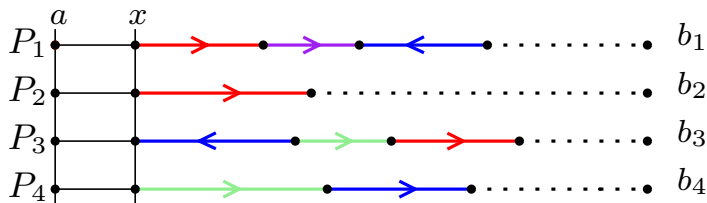
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Branched colouring

Idea : Generalize the previous observation recursively.

Take a set of paths from a to the vertices at distance D from a .

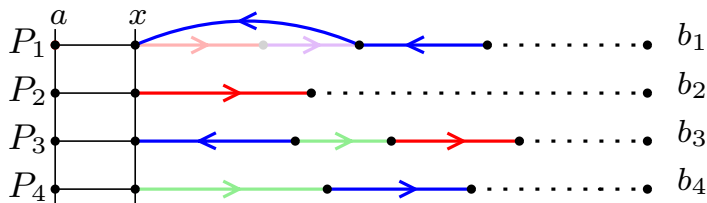


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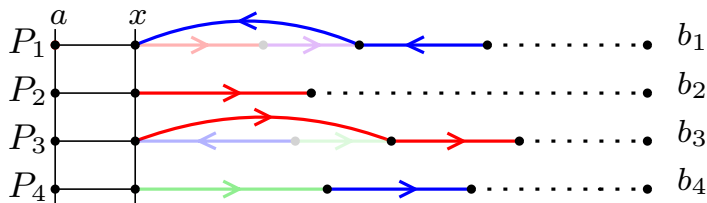


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) Apply recursively this process on each subset of paths independently.

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Structure of the paths at the end of the recursive process :

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Graphs vertex-coverable by k shortest paths

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A colour and a direction given to each base path.

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Colours-signs words are defined the same way as in the edge case.

Here : $! = ((c_1; +); (c_2; +); (c_3; +); (c_4; \quad));$

Bound for the vertex case

Good colouring Lemma works the same way.

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Branched colouring can be adapted to the vertex case, but $\frac{1}{k}$ $O(k)$ paths may share the same colours-signs word.

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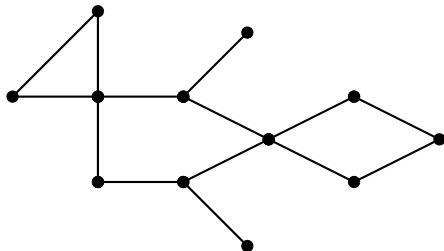
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Branched colouring can be adapted to the vertex case, but ~~the~~ $O(k)$ paths may share the same colours-signs word.

There is at most $g(k) = O(k \cdot 3^k)$ vertices at a given distance of a .

Algorithmic Consequences

Problems

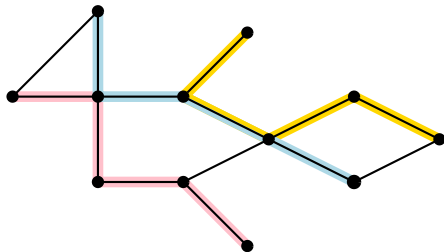


Isometric Path Cover (IPC)

Input : A graph G and an integer k .

Question : Does there exist a set of k shortest paths of G , such that each vertex of G belongs to at least one of the shortest paths?

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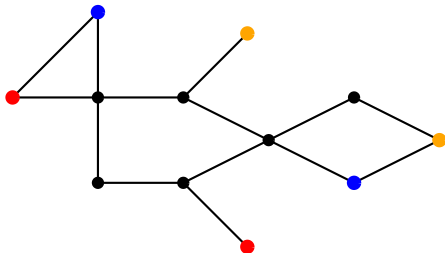


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with terminals



Isometric Path Cover with Terminals (IPC-WT)

Input : A graph G , and k pairs of vertices $(s_1; t_1); \dots; (s_k; t_k)$, the **terminals**.

Question : Does there exist a set of k shortest paths of G , the i th path being an s_i - t_i shortest path, such that each vertex of G belongs to at least one of the shortest paths?

What is known?

IPC is NP-Complete even on chordal graphs

[Chakraborty, Dailly, Das, Foucaud, Gahlawat, and Ghosh, 2022]

IPC is polynomial on block graphs

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Question

Are problems IPC and IPC-WT **FPT**? Or at least **XP**?

FPT : running time $f(k) n^{O(1)}$

XP : running time $n^{f(k)}$

Algorithmic consequences

Theorem 2

Problem IPC-WT is **FPT**, with running time $O(f(k) \cdot n)$.

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Yes-instances have bounded treewidth by Corollary 1,
Courcelle's theorem to solve this problem on bounded treewidth graphs.

Algorithmic consequences

Theorem 2

Problem IPC-WT is **FPT**, with running time $O(f(k) n)$.

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Corollary 2

Problem IPC is **XP** for parameter k .

Brute force : try all possible pairs of k terminals + FPT algorithm

Courcelle's Theorem

Theorem [Courcelle. 1990]

Every problem expressible in **monadic second order logic** (MSO_2) can be solved in $f(w) \cdot n$ time on graphs of treewidth at most w .

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Extended MSO_2 problem :

MSO_2 formula $\varphi(X_1; \dots; X_l)$ and an linear function $g(jX_1j; \dots; jX_lj)$

Find an assignation of $X_1; \dots; X_l$ that satisfies $\varphi(X_1; \dots; X_l)$ and maximize/minimize $g(jX_1j; \dots; jX_lj)$

Theorem [Arnborg, Lagergren, Seese. 1991]

Every problem expressible as an EMSO_2 problem can be solved in $f(w) \cdot n$ time on graphs of treewidth at most w .

Corollary 1

Graphs coverable with k shortest paths have treewidth bounded by $2g(k)$.

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FPT algorithm :

1. Compute a tree decomposition by BFS. If width $> 2g(k)$ return false.

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1. Compute a tree decomposition by BFS. If width $> 2g(k)$ return false.
2. Find $E_1; \dots; E_k$ minimizing $|E_1| + \dots + |E_k|$ and satisfying the MSO₂ formula :

$$\exists (E_1; \dots; E_k) = \exists V_1; \dots; V_k; \text{Cover}(V_1; \dots; V_k) \wedge \bigwedge_{1 \leq i \leq k} \text{Path}(V_i; E_i; s_i; t_i)$$

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3. If $|E_i| = \text{dist}(s_i; t_i)$ then answer true, else answer false.

Corollary 1

Graphs coverable with k shortest paths have treewidth bounded by $2g(k)$.

FPT algorithm :

1. Compute a tree decomposition by BFS. If width $> 2g(k)$ return false.
2. Find $E_1; \dots; E_k$ minimizing $|E_1| + \dots + |E_k|$ and satisfying the MSO₂ formula :

$$\exists (E_1; \dots; E_k) = \exists V_1; \dots; V_k; \text{Cover}(V_1; \dots; V_k) \wedge \bigwedge_{1 \leq i \leq k} \text{Path}(V_i; E_i; s_i; t_i)$$

3. If $|E_i| = \text{dist}(s_i; t_i)$ then answer true, else answer false.

) Can be easily generalized to edge covering & edge/vertex partitioning.

Conclusion

Conclusion

In graphs vertex/edge-coverable by k shortest paths, the number of vertices at same distance of a source is upper bounded by $g(k) = O(3^k)$.

Implies a $O(3^k)$ upper bound on treewidth.

Problem IPC-WT is FPT.

Problem IPC is XP.

Questions

Polynomial bound on the treewidth/pathwidth?

Is IPC (without terminals) FPT? W-hard?