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A linear algorithm to color i-triangulated graphs

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Abstract: We show that i-triangulated graphs can be colored in linear time by applying lexicographic breadth-first search (abbreviated LexBFS) and the greedy coloring algorithm.

Keywords: lexicographic breadth-first search, greedy algorithm, triangulated graph, i-triangulated graph

1 LexBFS and greedy algorithm

A graph is called *i-triangulated* if every odd cycle on five vertices or more contains at least two non-crossing chords. We are going to prove that the following algorithm (which works for triangulated graphs, see [3]) correctly colors any i-triangulated graph. As both steps can be performed in $O(m+n)$ time, this algorithm improves the complexity of the best previous algorithm:

Algorithm LexBFS-Col (input: $G = (V, E)$; output: (σ, c), where σ is an order, c is a coloring)

Step 1. apply lexicographic breadth-first search; call $\sigma : \{1, 2, \ldots, n\} \to V$ the resulting order.
Step 2. apply the greedy algorithm, using the converse order of σ; call $c : V \to \{1, 2, \ldots\}$ the resulting coloring.

The reader can find details on lexicographic breadth-first search (abbreviated LexBFS) and greedy coloring in [5, 3] and respectively [1]. We only give here brief descriptions of them. To this end, we assume that G is connected. The neighborhood of an arbitrary vertex x of G is denoted $N(x)$. If $W \subseteq V$, then $N_W(x) = N(x) \cap W$ and, for another subset $X \subseteq V$, $N_W(X) = \cup_{x \in X} N_W(x)$. For any other graph F ($F \neq G$), its vertex and edge sets are denoted respectively by $V(F)$ and $E(F)$.

The *lexicographic breadth-first search* has been proposed by Rose, Tarjan and Leuker [5] as an approach of triangulated graph recognition. We use all along the paper the description of LexBFS given by Golumbic [3]. Any vertex has a label which consists in a set of numbers listed in decreasing order. The labels may be compared using lexicographic order (i.e. dictionary order, using numbers instead of letters). Then the algorithm may be described as follows:

Algorithm LexBFS (input: G; output: an order $\sigma : \{1, 2, \ldots, n\} \to V$)

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assign the label ∅ to each vertex;
for i := n down to 1 do
    pick an unnumbered vertex v with largest label (in lexicographic order);
    σ(i) := v; {comment: this assigns to v the number i}
    for each unnumbered vertex t ∈ N(v) do
        add i to the label of t (at the end)
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Then we have (see [3]):

Lemma 1 *Let v, u, w such that $\sigma^{-1}(v) < \sigma^{-1}(u) < \sigma^{-1}(w)$ and $vw \in E, uw \notin E$. Then there exists a vertex $z \in N(u) - N(v)$ such that $\sigma^{-1}(w) < \sigma^{-1}(z)$.*

With the *greedy algorithm*, the graph is colored with colors 1, 2, 3, ... by considering the vertices in a given order and attributing, to each of them, the smallest color not yet given to one of its neighbours. It is an NP-complete problem to find, for an arbitrary graph, an order which yields an optimal coloring (although such an order always exists).

However, for i-triangulated graphs such an order may be find using LexBFS: in the rest of the paper, we will show that if the order obtained by LexBFS is $\sigma(1), \sigma(2), \dots, \sigma(n)$, then the greedy algorithm applied on the converse order $\sigma(n), \sigma(n-1), \dots, \sigma(1)$ performs an optimal coloring of the graph. This is equivalent to say that LexBFS-Col correctly colors G with minimum number of colors.

Before starting the proof, we note this small result (the proof is simple and may be found in [6]):

Lemma 2 *Let $G = (V, E)$ be an i-triangulated graph, B be a bipartite 2-connected subgraph of G and $x \in V - V(B)$. Then either x is adjacent to no vertex in B, or x is adjacent to all the vertices in B, or else $\{x\} \cup V(B)$ induces a bipartite graph.*

2 Main result

For any $v \in V$, denote $X^\sigma(v) = \{u \in N(v) \,|\, \sigma^{-1}(u) > \sigma^{-1}(v)\}$. Then we have:

Theorem 1 *Let (σ, c) be the result of Algorithm LexBFS-Col for the i-triangulated graph $G = (V, E)$. Then, for any vertex $z \in V$, any two nonadjacent vertices $x, y \in X^\sigma(z)$ satisfy $c(x) = c(y)$.*

Proof of Theorem 1. We use induction on the number of vertices of G. If $|V| = 1$ or $|V| = 2$, the theorem is trivially true. We assume it is true for every i-triangulated graph with less than $n = |V|$ vertices and we prove it for G. Denote $w = \sigma(n)$ (the first vertex picked by LexBFS).

By contradiction, if the contrary holds then there exists a vertex z and two nonadjacent neighbors of it (denoted x, y) such that $\sigma^{-1}(x) > \sigma^{-1}(y) > \sigma^{-1}(z)$ and $c(x) \neq c(y)$. We firstly prove that:

Claim 1 *Let u be an arbitrary vertex of $V - \{z\}$. Then $\sigma^{-1}(u) > \sigma^{-1}(z)$ and a chordless path $[u, v_1, ..., v_k, z]$ of G exists such that $\sigma^{-1}(u) > \sigma^{-1}(v_1) > \sigma^{-1}(v_2) > \cdots > \sigma^{-1}(v_k) > \sigma^{-1}(z)$.*

Proof of Claim 1. By contradiction, assume there exists u not satisfying the indicated condition and take the one with minimum $\sigma^{-1}(u)$. Then either $\sigma^{-1}(u) = 1$, or $\sigma^{-1}(z) < \sigma^{-1}(u)$ and for every q with $\sigma^{-1}(z) \leq \sigma^{-1}(q) < \sigma^{-1}(u)$, we have $qu \notin E$ (otherwise the choice of u is contradicted).

We apply the algorithm LexBFS on $G - u$ such that any time we have to pick a vertex (from the set of vertices with largest label), we pick q with $\sigma(q)$ maximum. Denote by σ_1 the order obtained in this way and notice that σ_1 is exactly the order σ where u has been removed (to be rigourous, some indices must be shifted when $\sigma^{-1}(z) < \sigma^{-1}(u)$). This is due to the fact that in the last step of the algorithm LexBFS (with v replaced by u) no unnumbered vertex $t \in N(u)$ exists. So, if we achieve the application of LexBFS-Col on $G - u$ using the greedy algorithm, we find a coloring c_1 of $G - u$ such that $c_1(x) = c(x), c_1(y) = c(y), c_1(z) = c(z)$. Then Theorem 1 is contradicted for $G - u$.$\blacksquare$

Obviously, Claim 1 implies that $\sigma^{-1}(z) = 1$. Now, we define a partition $H_0, H_1, \dots, H_p, H'_p$ of V as follows. Set $H_0 = \{w\}$. By induction, assume that $H_i \subseteq V - \{z\}$ is already defined, and put

$H'_i = \{h' \in V \mid \text{for all}\, h \in H_i, \sigma^{-1}(h) > \sigma^{-1}(h')\}$.

Furthermore, let us partition H_i in three subsets:

$T_i = H_i \cap N(z)$, $U_i = \{h \in H_i - T_i \,| N_{H'_i}(h) \neq \emptyset\}$, and $V_i = H_i - (T_i \cup U_i)$.

Now, we can define $H_{i+1} = N_{H'_i}(U_i)$. Assume this construction is repeated as long as the set H_{i+1} resulting at each step is non-empty (equivalently, as long as $U_i \neq \emptyset$) and let us notice that:

Remark 1 Claim 1 implies that for all $h' \in H'_i$ $(0 \leq i \leq p)$, a chordless path $[h', a_1, ..., a_l, z]$ joining h' to z in H'_i exists such that $\sigma^{-1}(h') > \sigma^{-1}(a_1) > \sigma^{-1}(a_2) > \ldots > \sigma^{-1}(a_l) > \sigma^{-1}(z)$.

Let us say that a set $R \subseteq V$ is *compact* if any $u \in V - R$ either satisfies $\sigma^{-1}(r) > \sigma^{-1}(u)$ for any $r \in R$, or satisfies $\sigma^{-1}(r) < \sigma^{-1}(u)$ for any $r \in R$. We can now prove that for an arbitrary i (see Fig. 1.):

Claim 2 *The following properties hold for H_{i+1} (such that $i \geq 0$ and $H_{i+1} \neq \emptyset$):*
a) it is compact and $H_{i+1} \subseteq H'_i, H'_{i+1} \subset H'_i$;
b) no $v \in V$ satisfies $\sigma^{-1}(v) > \sigma^{-1}(h')$ for any $h' \in H_{i+1}$, and $\sigma^{-1}(v) < \sigma^{-1}(h)$ for any $h \in H_i$;
c) every $h' \in H'_{i+1}$ has $N(h') \subseteq H'_{i+1} \cup H_{i+1} \cup T_i \cup T_{i-1} \cup \ldots \cup T_0$.

Proof of Claim 2. We have nothing to prove for *c)*, it is an immediate consequence of the definitions above. For *a)* and *b)* we use induction on $i \geq 0$. For $i = 0$, we have $H_0 = \{w\}$, $H'_0 = V - \{w\}$, $U_0 = \{w\}$ (we have assumed $H_1 \neq \emptyset$, i.e. $U_0 \neq \emptyset$). Then $H_1 = N(w)$ and this is a compact set. There is nothing to prove for the second part of *a)* and for *b)*.

Assume now that *a), b)* are true for each $j < i$ and let us prove them for i. For an arbitrary vertex y, we use the notation $label(y)$ to indicate its label at a specified moment of LexBFS.

Proof of a): By definition, $H_{i+1} \subseteq H'_i, H'_{i+1} \subset H'_i$. If, by contradiction, H_{i+1} is not compact, there exist vertices $a, c \in H_{i+1}$ and $b \notin H_{i+1}$ such that $\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$. Then $b \in H'_i$ (since $\sigma^{-1}(b) < \sigma^{-1}(c)$ and $c \in H'_i$), and by *c)* for i (instead of $i+1$) we have $N(b) \subseteq H'_i \cup H_i \cup T_{i-1} \cup T_{i-2} \cup \ldots \cup T_1 \cup T_0$. Moreover, $b \notin H_{i+1}$, so b is adjacent to no vertex in U_i, consequently $N(b) \subseteq H'_i \cup T_i \cup T_{i-1} \cup T_{i-2} \cup \ldots \cup T_1 \cup T_0$.

Now, when b is picked by LexBFS (and a is *not* picked), we have $label(a) \leq label(b)$, so:

(P) there exists $t^0 \in T_i \cup T_{i-1} \cup \ldots \cup T_0$ such that $t^0 b \in E, t^0 a \notin E$ and $\sigma^{-1}(t^0) > \sigma^{-1}(u)$ for every $u \in U_i$ which is adjacent to a.

Indeed, if some $t^0 \in T_{i-1} \cup \ldots \cup T_0$ exists such that $t^0 b \in E, t^0 a \notin E$, the property is obviously true. Otherwise, $N_{T_{i-1} \cup \ldots \cup T_0}(a) = N_{T_{i-1} \cup \ldots \cup T_0}(b)$, and we cannot have $N_{T_i}(a) \supseteq N_{T_i}(b)$ since then (using some neighbor u of a in U_i) we would have $label(a) > label(b)$. Thus there exists some $t^0 \in (N(b) - N(a)) \cap T_i$ and we take the one with largest $\sigma^{-1}(t^0)$. We also consider $u \in N_{U_i}(a)$ (which is not in $N(b)$ since b has no neighbor in U_i) such that $\sigma^{-1}(u)$ is maximum. If $\sigma^{-1}(u) < \sigma^{-1}(t^0)$, then we are done. Otherwise $label(a) > label(b)$, a contradiction.

So (P) is proved. But then at the moment where a was picked by LexBFS (and z was not picked) we had $label(z) > label(a)$, since z is adjacent to every vertex in $T_i \cup T_{i-1} \cup \ldots \cup T_0$, so $zt^0 \in E$. We have a contradiction.

Proof of b): By contradiction, if such a vertex v exists then $v \in H'_i$ (by hypothesis) and the reasoning we made before for b is valid (taking some $a \in H_{i+1}$) to obtain a contradiction. ∎

Remark 2 By the preceding claim (affirmations *a)* and *b)*), the sets $H_0, H_1, \ldots, H_i, H'_i$ form a partition of V in compact sets.

Since for any i such that $U_i \neq \emptyset$ we have $H_{i+1} \neq \emptyset$, the set $H_0 \cup H_1 \cup \ldots \cup H_{i+1}$ grows. As the graph is finite, there will exist some p (as small as possible) such that $H_{p+1} = \emptyset$, i.e. $U_p = \emptyset$.

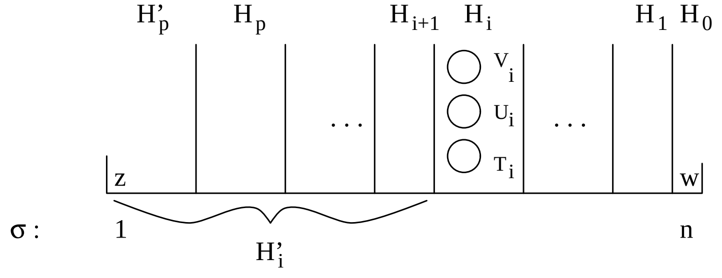

Fig. 1. The sets H_i, H'_i $(i = 0, 1, \ldots, p)$

Claim 3 *For all $i \in \{0, ..., p\}$ the following properties hold :*

(A_i) *for any $h_1, h_2 \in H_i$, there exists a path $[h_1, v_1, ..., v_{2k+1}, h_2]$ with at most one chord (which is necessarily short) and such that $\{v_1, ..., v_{2k+1}\} \subseteq V - (H_i \cup H'_i \cup N_V(H'_i))$. Moreover, if $h_1 h_2 \in E$ then $k = 0$.*

(B_i) *for all $v \in V - (H'_i \cup H_i)$, if v is adjacent to a vertex of H'_i then it is adjacent to all the vertices of $H'_i \cup H_i$.*

(C_i) *for any $h'_1, h'_2 \in H'_i$ such that $h'_1 h'_2 \in E$ and $\sigma^{-1}(h'_1) > \sigma^{-1}(h'_2)$, we have $N_{H_i}(h'_2) \subseteq N_{H_i}(h'_1)$ and every vertex of $N_{H_i}(h'_1) - N_{H_i}(h'_2)$ is adjacent to all the vertices of $N_{H_i}(h'_2)$.*

(D_i) *for every $t \in T_i$, t is adjacent to all the vertices of H'_i.*

(E_i) *for every $u \in U_i$, u is adjacent to all the vertices of T_i.*

(F_i) *if $U_i \neq \emptyset$ then T_i is a clique (or is empty).*

Proof of Claim 3. We use induction on i. Clearly (A_0), (B_0), (C_0), (D_0), (E_0) and (F_0) hold. Now, assume that for $0 \leq j \leq i-1$ the claim holds. By Claim 2, $H'_i = H_{i+1} \cup H'_{i+1}$ for each i.

Proof of (A_i): Let $h_1, h_2 \in H_i$. Since $H_i = N_{H'_{i-1}}(U_{i-1})$, there exist $u_1, u_2 \in U_{i-1}$ such that $u_1 h_1 \in E$ and $u_2 h_2 \in E$. If $u_1 = u_2$ then there is nothing to prove. Otherwise, because of $H'_i \cap N(U_{i-1}) = \emptyset$, for every $h' \in H'_i$ we have $h' u_1 \notin E$ and $h' u_2 \notin E$. By (A_{i-1}), there exists a path $P = [u_1, v_1, ..., v_{2k+1}, u_2]$ with at most one short chord and such that $\{v_1, ..., v_{2k+1}\} \subseteq V - (H_{i-1} \cup H'_{i-1} \cup N(H'_{i-1}))$. Thus, for every $j \in \{1, ..., 2k+1\}$, $h_1 v_j \notin E$ and $h_2 v_j \notin E$. Now, if $h_1 h_2 \in E$ then $P \cup \{h_1, h_2\}$ forms an odd cycle with at most one chord (which is necessarily short), a contradiction with the fact that G is an i-triangulated graph. Otherwise, $[h_1, u_1, v_1, ..., v_{2k+1}, u_2, h_2]$ is a path with at most one short chord and such that $\{u_1, v_1, ..., v_{2k+1}, u_2\} \subseteq V - (H_i \cup H'_i \cup N(H'_i))$.

Proof of (B_i): Let $v \in V - (H'_i \cup H_i)$ such that at least one vertex (noted h') of H'_i is adjacent to v. If $v \in V - (H'_{i-1} \cup H_{i-1})$ then by (B_{i-1}) we know that v is adjacent to every vertex of H'_{i-1} (which is equal to $H_i \cup H'_i$). Otherwise $v \in H_{i-1}$ (which is equal to $V_{i-1} \cup U_{i-1} \cup T_{i-1}$). Since $N(V_{i-1} \cup U_{i-1}) \cap H'_i = \emptyset$, we have $v \notin (V_{i-1} \cup U_{i-1})$. Then $v \in T_{i-1}$ and by (D_{i-1}) we obtain that v is adjacent to every vertex of $H'_{i-1} = H_i \cup H'_i$.

Proof of (C_i): Let h'_1 and h'_2 be two adjacent vertices of H'_i such that $\sigma^{-1}(h'_1) > \sigma^{-1}(h'_2)$.

If $N_{H_i}(h'_2) \not\subseteq N_{H_i}(h'_1)$ then there exists a vertex $u_2 \in H_i$ adjacent to h'_2 and nonadjacent to h'_1. Since $\sigma^{-1}(h'_2) < \sigma^{-1}(h'_1) < \sigma^{-1}(u_2)$, by Lemma 1 there exists a vertex $u_1 \in N(h'_1) - N(h'_2)$ such that $\sigma^{-1}(u_1) > \sigma^{-1}(u_2)$. Then by (B_i), $u_1 \in H_i$ and (A_i) implies that there exists a path $P = [u_1, v_1, ..., v_{2k+1}, u_2]$ with at most one short chord and such that $\{v_1, ..., v_{2k+1}\} \cap N(\{h'_1, h'_2\}) = \emptyset$. Then, $\{h'_1, h'_2\} \cup P$ forms an odd cycle with at most one short chord, which is a contradiction.

Let us suppose that there exist $h_1 \in N_{H_i}(h'_1) - N_{H_i}(h'_2)$ and $h_2 \in N_{H_i}(h'_2)$ such that $h_1 h_2 \notin E$. By (A_i), there exists a path $P = [v_0 = h_1, v_1, ..., v_{2k+1}, h_2 = v_{2k+2}]$ with at most one chord (which is necessarily short and is denoted $v_j v_{j+2}$, with $0 \leq j \leq 2k$) and such that $\{v_1, ..., v_{2k+1}\} \cap N(\{h'_1, h'_2\}) = \emptyset$. If P has a short chord, then $\{h'_1\} \cup P - \{v_{j+1}\}$ induce a bad odd cycle. Otherwise $\{h'_1, h'_2\} \cup P$ gives a bad odd cycle.

Proof of (D_i): Let t be a vertex of T_i and h' be a vertex of H'_i. By Remark 1 a chordless path $[h', a_1, ..., a_l, z]$ exists in H'_i such that $\sigma^{-1}(h') > \sigma^{-1}(a_1) > \ldots > \sigma^{-1}(a_l) > \sigma^{-1}(z)$. We know that $zt \in E$. Then, by (C_i), applied several times, we have $ta_l, ta_{l-1}, \ldots, ta_1, th' \in E$.

Proof of (E_i): Let $u \in U_i$. We recall that $uz \notin E$ and there exists $h'' \in H'_i$ such that $uh'' \in E$ (by the definition of U_i). Thus, by Remark 1, a chordless path $[b_0 = h'', b_1, ..., b_s, b_{s+1} = z]$ exists in H'_i such that $\sigma^{-1}(h'') > \sigma^{-1}(b_1) > \ldots > \sigma^{-1}(b_s) > \sigma^{-1}(z)$. By (D_i), t is adjacent to every vertex in $\{b_0, b_1, ..., b_s, b_{s+1}\}$. Let i $(0 \leq i \leq k)$ be the largest index such that $b_i u \in E$ and $b_{i+1} u \notin E$ (such an index must exist since $ub_{s+1} \notin E$). Since (C_i) is true and $tb_i, tb_{i+1} \in E$, u is adjacent to t.

Proof of (F_i): If (F_i) is false then two nonadjacent vertices (denoted t_1 and t_2) of T_i exist and $U_i \neq \emptyset$. Let u be a vertex of U_i. We recall that $t_1, t_2 \in N(z)$ and u is not adjacent to z. Moreover, by (E_i) we have $ut_1, ut_2 \in E$. Since (A_i) is true, a vertex v_1 nonadjacent to z and adjacent to u and t_1 exists. In both cases when $t_2 v_1 \in E$ or $t_2 v_1 \notin E$, the set $\{z, t_1, u, t_2, v_1\}$ induces a bad odd cycle.∎

We continue the proof of Theorem 1. Recall that, by contradiction with the conclusion of Theorem 1, we have supposed that there exist vertices $x, y \in X^\sigma(z)$ such that $xy \notin E$ and $c(x) \neq c(y)$. We now try to find a contradiction.

By Claim 3 (statements (D_j) and (F_j)) and since $U_j \neq \emptyset$ $(0 \leq j \leq p-1)$, the set $T_0 \cup T_1 \cup \ldots \cup T_{p-1}$ is a clique (notice that T_p is not necessarily a clique since $U_p = \emptyset$ and the affirmation (F_p) cannot be used). By *c)* in Claim 2, we have that $N(z) \cap (V - H'_p) = T_0 \cup \ldots \cup T_{p-1} \cup T_p$. Moreover, since (D_j) holds for $0 \leq j \leq p-1$, every vertex of $T_0 \cup \cdots \cup T_{p-1}$ is adjacent to all the vertices of H'_{p-1} (which is equal to $H'_p \cup H_p$). Then we have that $\{x, y\} \subset H'_p \cup T_p$. Moreover, by (D_p) every vertex of T_p is adjacent to all the vertices of H'_p. So x, y are simultaneously in H'_p or in T_p.

Case 1. $x, y \in H'_p$. By (B_p) and since $U_p = \emptyset$, for every $\{h'_1, h'_2\} \subseteq H'_p$ we have $N_{V-H'_p}(h'_1) = N_{V-H'_p}(h'_2)$. Thus, all the vertices in H'_p have *exactly* the same neighbors in $V - H'_p$. Consequently, the behavior of the algorithm LexBFS for G, when restricted to the subgraph $[H'_p]_G$ induced by H'_p, is not influenced by the vertices in $V - H'_p$; locally, it is the same as if LexBFS was applied directly on the subgraph $[H'_p]_G$. The same holds for LexBFS-Col: locally, its behaviour is the same as if it was applied directly on the subgraph $[H'_p]_G$ (only the set of colors will be changed, since in G there exist some forbidden colors for the vertices in H'_p). Thus, by induction hypothesis, $[H'_p]_G$ (which is strictly smaller than G) satisfies Theorem 1. But then the colors of x, y must be identical in $[H'_p]_G$ and thus identical in G (since the behaviour of the algorithm is the same, only the set of colors is changed), a contradiction to our assumption that $c(x) \neq c(y)$.

Case 2 : $x, y \in T_p$. Then $[x, z, y]$ is a chordless path joining x to y such that $z \in H'_p$. A similar property holds for any index j, $0 \leq j \leq p$:

(Q_j) There exist two nonadjacent vertices of H_{p-j} (denoted x_j and y_j) such that $c(x_j) \neq c(y_j)$ and x_j, y_j are joined by a chordless path $[x_j, u_1, ..., u_l, y_j]$ with $\{u_1, ..., u_l\} \subseteq H'_{p-j}$.

To prove this, we use induction on j. With the notation $x_0 = x$ and $y_0 = y$, the affirmation (Q_0) holds. Assume that (Q_{j-1}) is true. Then there exist $x_{j-1}, y_{j-1} \in H_{p-j+1}$ such that $c(x_{j-1}) \neq c(y_{j-1})$. Moreover, there exists a chordless path $[x_{j-1}, u_1, ..., u_h, y_{j-1}]$ such that $\{u_1, ..., u_h\} \subseteq H'_{p-j+1}$. We can suppose without loss of generality that $c(x_{j-1}) > c(y_{j-1})$. Thus, as G was colored using the greedy algorithm, a vertex x_j must exist such that $x_j x_{j-1} \in E$, $\sigma^{-1}(x_j) > \sigma^{-1}(x_{j-1})$ and $c(x_j) = c(y_{j-1})$. Then $x_j \in H_{p-j+1} \cup U_{p-j} \cup T_{p-j} \cup T_{p-j-1} \cup \ldots \cup T_0$. By (B_{p-j}) and (D_{p-j}), we know that every vertex of $T_{p-j} \cup \ldots \cup T_0$ is adjacent to all the vertices of $H_{p-j+1} \cup H'_{p-j+1}$. Since $c(x_j) = c(y_{j-1})$, we have $x_j y_{j-1} \notin E$. Thus $x_j \notin T_{p-j} \cup \ldots \cup T_0$, so $x_j \in H_{p-j+1} \cup U_{p-j}$.

By (A_{p-j+1}), a *chordless* path $[x_{j-1}, v_1, ..., v_r, y_{j-1}]$ joining x_{j-1} to y_{j-1} exists such that $\{v_1, ..., v_r\} \subseteq V - (H_{p-j+1} \cup H'_{p-j+1} \cup N_V(H'_{p-j+1}))$. Since $v_1 x_{j-1} \in E$ and $v_1 y_{j-1} \notin E$, we cannot have $v \in H_{p-j-1} \cup H_{p-j-2} \cup \ldots \cup H_0$ (by (B_{p-j})). Then $v_1 \in H_{p-j}$. Observe that $C = [x_{j-1}, v_1, ..., v_r, y_{j-1}, u_h, ..., u_1, x_{j-1}]$ is a chordless cycle.

Recall that we have $x_j \in H_{p-j+1} \cup U_{p-j}$. We prove that $x_j \notin H_{p-j+1}$. Indeed, if the contrary holds then by (C_{p-j}) we have $x_j v_1 \in E$. Now, by Lemma 2 and since x_j is adjacent to at least two consecutive vertices of C, x_j must be adjacent to every vertex of C, which is a contradiction with the fact that $x_j y_{j-1} \notin E$. Hence, $x_j \in U_{p-j}$. Since $y_{j-1} \in H_{p-j+1}$, a vertex $y_j \in U_{p-j}$ exists such that $y_j y_{j-1} \in E$ (so $y_j \neq x_j$). Now, $c(x_j) = c(y_{j-1})$ implies that $c(y_j) \neq c(x_j)$. We recall that $U_{p-j} \cap N(H'_{p-j+1}) = \emptyset$ (by the definition of H_{p-j+1}). Two cases can occur:

• If $y_j x_{j-1} \notin E$, then $P = [x_j, x_{j-1}, u_1, ..., u_h, y_{j-1}, y_j]$ is a chordless path such that $\{x_{j-1}, u_1, ..., u_h, y_{j-1}\} \subseteq H'_{p-j}$. To finish the proof in this case we only have to show that $x_j y_j \notin E$. By contradiction, if $x_j y_j \in E$ then $C' = [x_j, x_{j-1}, u_1, ..., u_h, y_{j-1}, y_j, x_j]$ is a chordless cycle with at least four vertices (which must be even). By (A_{p-j}), there exists a vertex $v'_1 \in V - (H_{p-j} \cup H'_{p-j} \cup N_V(H'_{p-j}))$ which is adjacent to x_j and y_j. Thus, $C' \cup \{v'_1\}$ gives a contradiction with Lemma 2.

• If $y_j x_{j-1} \in E$, then $P' = [x_j, x_{j-1}, y_j]$ is a chordless path such that $x_{j-1} \in H'_{p-j}$. To finish the proof, we need to show that $x_j y_j \notin E$. In the contrary case, $C'' = [y_j, x_{j-1}, u_1, ..., u_h, y_{j-1}, y_j]$ is a chordless cycle with at least four vertices which must be even. As x_j is adjacent to two consecutive vertices on C'', by (A_{p-j}) we should have $x_j y_{j-1} \in E$, a contradiction.

Then (Q_j) holds for $0 \leq j \leq p$ and (Q_p) implies $|H_0| \geq 2$, a contradiction with $H_0 = \{w\}$.∎

3 Some corollaries

The order obtained using LexBFS on an i-triangulated graph is a special order, as shown by the following result:

Corollary 1 *Let $G = (V, E)$ be an i-triangulated graph and σ be an order obtained using LexBFS. Then, for any $s \in V$, $X^\sigma(s)$ is a complete multi-partite graph.*

Proof of Corollary 1. If there exists some s for which $X^\sigma(s)$ is not a complete multi-partite graph, then three vertices $v, x, y \in X^\sigma(s)$ may be found such that $xy \in E$, $xv \notin E$ and $yv \notin E$. Apply LexBFS-Col, using in Step 1 the algorithm LexBFS which gives σ. Then, Theorem 1 implies that $c(x) = c(v)$ and $c(y) = c(v)$. Thus $c(x) = c(y)$, a contradiction. ∎

This property is similar to the one of triangulated graphs (see [3]), where for any vertex $s \in V$ the set $X^\sigma(s)$ induces a clique. A triangulated graph (with at least two vertices) also has the property that it contains two *simplicial* vertices (a vertex is said *simplicial* if its neighborhood induces a clique). For i-triangulated graphs, we have:

Corollary 2 *Any i-triangulated graph $G = (V, E)$ (with $|V| \geq 2$) contains at least two vertices x_1, x_2 such that $N(x_i)$ induces a complete multi-partite graph.*

Proof of Corollary 2. Apply LexBFS starting with an arbitrary vertex w (denote by σ the resulting order) and then apply LexBFS again starting with the vertex $\sigma(1)$ (denote by σ_1 the resulting order). Then $\sigma(1), \sigma_1(1)$ are the desired vertices. ∎

Finally, let us prove that LexBFS-Col correctly colors i-triangulated graphs.

Corollary 3 *Let $G = (V, E)$ be an i-triangulated graph, and let (σ, c) be the result of LexBFS-Col when it is applied to G. Then c uses minimum number of colors.*

Proof of Corollary 3. It is sufficient to show that there exists a clique in G which has r vertices, where r is the number of colors used by LexBFS-Col. Consider a vertex $s \in V$ such that $c(s) = r$. Then $X^\sigma(s)$ induces a complete multi-partite graph, and all the colors $1, 2, \ldots, r-1$ appear in $X^\sigma(s)$. By Theorem 1, two different colors cannot appear in the same stable set of the complete multi-partite graph induced by $X^\sigma(s)$. Consequently, any stable set corresponds to a color, so there exists exactly $r-1$ stable sets in $X^\sigma(s)$. Now, it is sufficient to pick a vertex in each stable set and to add s in order to obtain a clique of size r in G. ∎

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