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About the Strong Perfect Graph Conjecture on circular partitionable graphs.

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About the Strong Perfect Graph Conjecture on circular partitionable graphs.

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Abstract

There is a construction due to Chvátal, Graham, Perold and Whitesides for making partitionable graphs with circular symetry. It is known that this construction doesn't produce any counter-example to Berge's conjecture.

Grinstead conjectured that any circular partitionable graph is generated by this construction. This question is still open. Bacsó, Boros, Gurvich, Maffray and Preissmann proved that every minimally imperfect graph with circular symmetry must have an odd number of vertices.

Here we prove that every minimally imperfect graph with circular symmetry doesn't have two distinct maximum cliques sharing at least one half of their vertices and two distinct maximum stable sets sharing at least one half of their vertices.

1 The circular partitionable graphs defined by Chvátal, Graham, Perold and Whitesides.

In 1960, Claude Berge introduced the notion of perfect graphs: a graph is *perfect* if for every induced subgraph H, the chromatic number of H doesn't exceed the maximum number of pairwise adjacent vertices in H. Berge conjectured that perfect graphs are graphs with no induced odd holes and their complements (a *hole* is a chordless cycle with at least four vertices), or equivalently that *minimal imperfect graphs* are odd holes and their complements.

A graph G is *partitionable* if it has $\alpha\omega+1$ vertices and for every vertex v of G, the induced subgraph $G-v$ admits a partition in α cliques of cardinality ω and also admits a partition in ω stable sets of cardinality α.

Lovász and Padberg proved that every minimally imperfect graph is partitionable. Hence an approach to Berge's conjecture is to find out the minimally imperfect graphs in some classes of partitionable graphs.

This paper deals with a special class of partitionable graphs: the class of partitionable graphs with circular symetry.

1.1 Near-factorizations of the cyclic group and circular partitionable graphs.

Let A and B be two subsets of Z_n (resp. N), the subset $A+B$ is equal to $\{a+b,\ a\in A,\ b\in B\}$ where $+$ is the sum operator of Z_n (resp. N). Hence $A+B$ depends on the context due to the nature of A and B.

Two subsets A and B of Z_n are said to form a *near-factorization* of Z_n iff

$$n = |A| * |B| + 1 \text{ } and \text{ } \exists u, A + B = Z_n \backslash \{u\}$$

If (A, B) is a near-factorization with $min\,(|A|, |B|) \geq 2$, *G(A,B)* is the Cayley graph with respect to $A - A$, that is with vertex set Z_n and edge set $\{e_{ij},\ i - j \in (A - A)\backslash\{0\}\}$. Then G(A,B) is a circular normalized partitionable graph, with maximum cliques $\{A + i,\ i \in Z_n\}$ and maximum stable sets $\{B + i,\ i \in Z_n\}$. Conversly, if G is any circular partitionable graph, C is an $\omega-$clique of G, S is an $\alpha-$stable of G, then (C, S) is a near-factorization of Z_n such that G(C,S) is the normalized graph of G.

See figures 1,2,3 for examples of near-factorizations and graphs G(A,B).

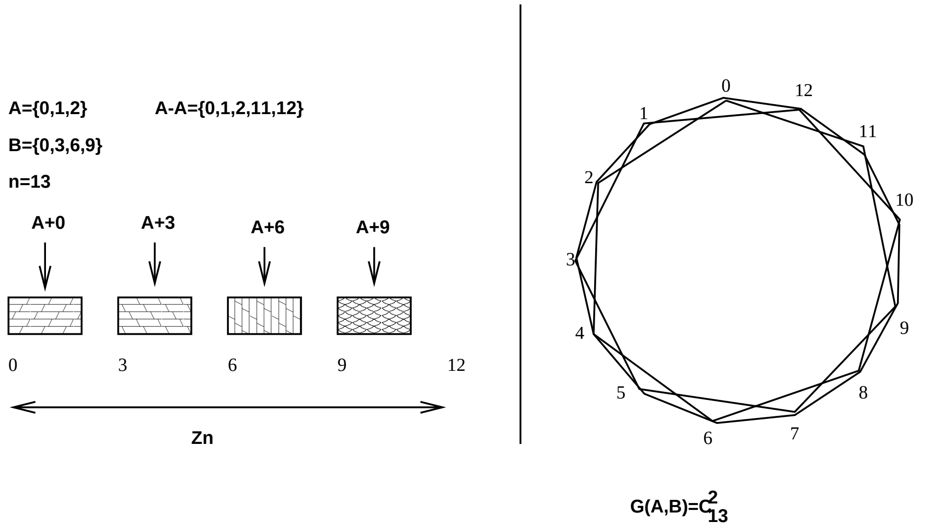

Figure 1: $A = \{0, 1, 2\},\ B = \{0, 3, 6, 9\}$

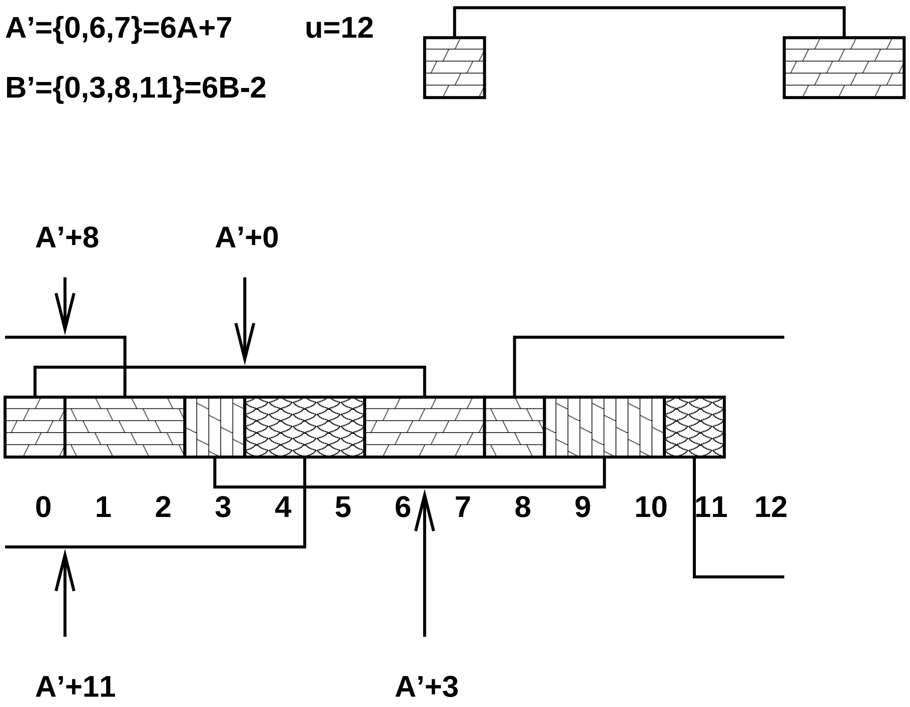

Figure 2: $A' = \{0, 6, 7\}$, $B' = \{0, 3, 8, 11\}$

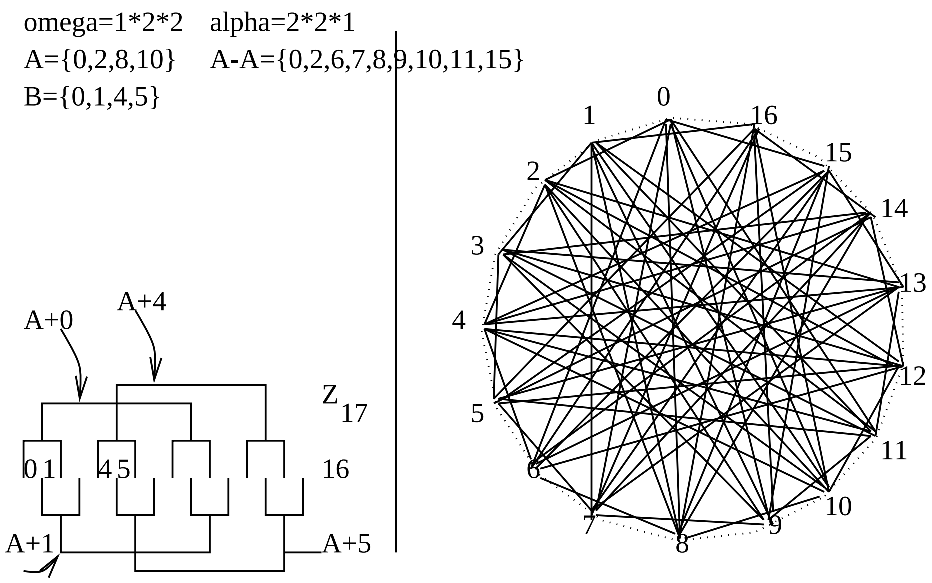

Figure 3: Example given by Chvátal, Graham, Perold and Whitesides

1.2 Generating a class of near-factorizations : definition of the CGPW graphs.

Two subsets A_1 and B_1 of N are said to form an *near-factorization in integers* iff $A_1 + B_1 = [0..|A_1| * |B_1| - 1]$. Obviously, a near-factorization in integers induces a near-factorization.

Let (A_1, B_1) be a near-factorization in integers such that $A_1 + B_1 = [0..n_1 - 2]$. Let k, k' be any positive integer.

One may obtain a near-factorization in integers (A_2, B_2) (see figure 4) such that $A_2 + B_2 = [0, n_2 - 2]$ with

$$n_2 := (|A_1| * k) * (|B_1| * k') + 1$$

by defining:

$$A_2 := A_1 + (n_1 - 1) * [0, k - 1] \; and \; B_2 := B_1 + (n_1 - 1) * k * [0, k' - 1]$$

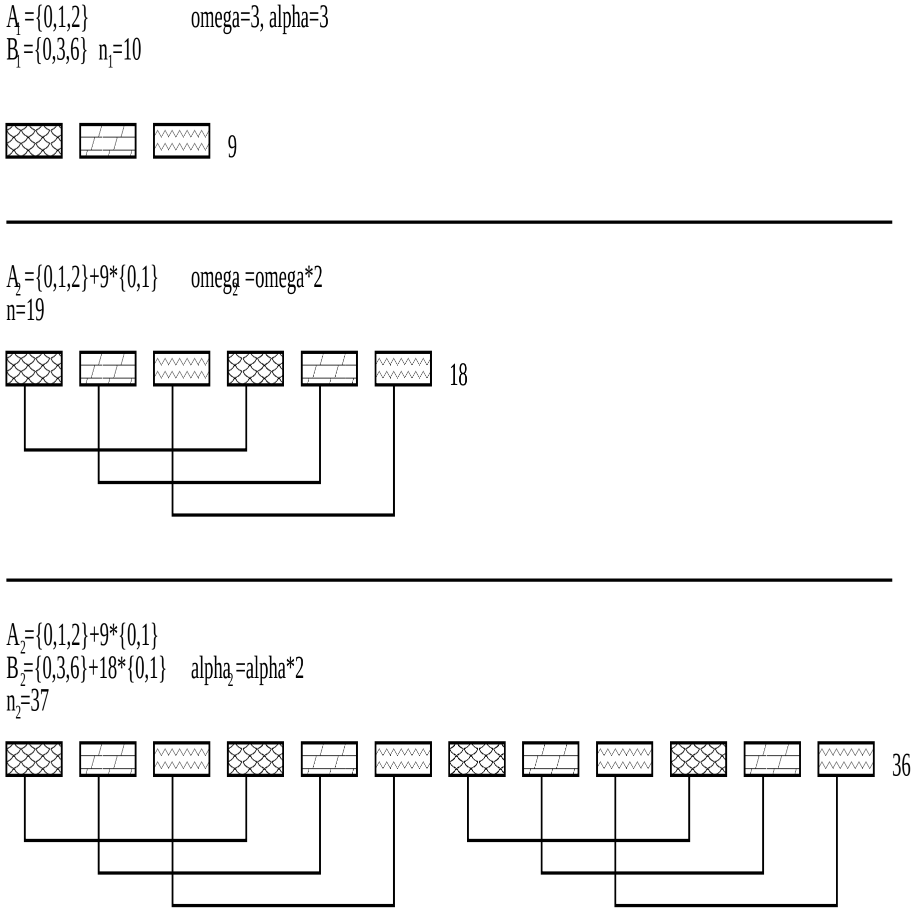

Figure 4: The method with: $\omega = 3,\ \alpha = 3\,; \omega_2 = 3 * 2,\ \alpha_2 = 3 * 2$

This method was introduced by V. Chvátal, R.L. Graham, A.F. Perold and S.H. Whitesides in 1979 [3].

A *CGPW* graph is a graph $G(A, B)$ where (A, B) is obtained with a finite number of applications of this method (starting from a *basic factorization*, that is $A = [0, |A| - 1]$: basic factorizations with $min(|A|, |B|) \geq 2$ are associated with *webs*).

Explicitly, the CGPW graph G given by $2p$ positive integers $a_1, ..., a_{2p}$ is constructed this way :

- take $A_1 = [0, a_1 - 1]$ and $B_1 = a_1 * [0, a_2 - 1]$.
- take $k = a_3$ and $k' = a_4$ then calculate A_2 and B_2.
- take $k = a_5$ and $k' = a_6$ then calculate A_3 and B_3 starting from A_2 and B_2
- ...
- until $k = a_{2p-1}$ and $k' = a_{2p}$.

G is $G(A_p, B_p)$ and is denoted $C[a_1, ..., a_{2p}]$.

In 1984, C.M. Grinstead [6] proved that any CGPW graph contains either a hole or an antihole.

In 1997, G. Bacso, E. Boros, V. Gurvich, F. Maffray and M. Preissmann [1] proved that the Strong Perfect Graph Conjecture holds for the variant of CGPW graphs. A *variant of a CGPW graph* G is a graph having the same $\omega-$cliques, the same $\alpha-$stables that G and the same number of vertices.

They proved that any variant of a CGPW graph admits a small transversal or if $\alpha = \omega = 2^k$ contains an induced C_5.

In 1984, C.M. Grinstead [6] asked the question whether every circular partitionable is a CGPW graph. Hence a positive answer would imply the Strong Perfect Graph Conjecture for the class of circular partitionable graphs.

He checked all the cases with the help of a computer case for $n < 50$ and for $\alpha = 6$ & $\omega = 10$ and found no counter-example to this conjecture.

In 1997, G. Bacso, E. Boros, V. Gurvich, F. Maffray and M. Preissmann [1] proved that this conjecture holds for any circular partitionable graph with $min(\alpha, \omega) \leq 5$.

1.3 Intersection degree of $\omega-$cliques of CGPW graphs.

Let G be a circular partitionable graph with $\omega-$clique A and $\alpha-$stable set B. Let $INT(A)$ (resp. $INT(B)$) be the maximum of the cardinal of the intersection of A (resp. B) with the others $\omega-$cliques (resp. $\alpha-$ stable-sets), that is:

$$INT(A) := max\{|A \cap A + i|, i \in [1, n - 1]\}$$

$$INT(B) := max\{|B \cap B + i|, i \in [1.n - 1]\}$$

Lemma 1 *Let G be a CGPW graph $C[a_1, ..., a_{2p}]$ with $\omega = a_1 * a_3 ... * a_{2k+1}$. Then*

$$INT(A) \geq \left(1 - \frac{1}{max\{a_i\}_{i=1,3,...,2k+1}\{a_i\}}\right) * \omega \geq \frac{\omega}{2}$$

Proof.

In [6], C.M. Grinstead proved that $C[a_1, ..., a_{2p}]$ is isomorphic to $C[a_3, ..., a_{2p}, a_1, a_2]$. Hence, we may suppose that $a_1 = max\{a_i\}_{i=1,3,...,2k+1}$ Let A_1 be an $\omega-$clique of $C[a_1, a_2]$. Then $INT(A_1) = a_1 - 1$. Since A_2 is made of k copys of A_1, we have: $INT(A_2) \geq (k * INT(A_1))$. And so on... So, if Q is an $\omega-$clique of $C[a_1, ..., a_{2p}]$ then $INT(Q) \geq (a_3 * ... * a_{2p-1}) * (a_1 - 1) = \left(1 - \frac{1}{a_1}\right) * \omega$. □

Hence, if Grinstead's conjecture holds then any circular partitionable graph would have two $\omega-$cliques intersecting in at least $\frac{\omega}{2}$ vertices. In section 2, we show that this would be enough to prove the Strong Perfect Graph Conjecture for circular partitionable graphs.

1.4 Webs are circular partitionable graphs having an $\alpha-$critical edge.

An edge e of a graph G is said to be an $\alpha - critical\ edge$ iff $\alpha(G-e) > \alpha(G)$. Similarly, a non-edge e' is said to be $co - critical$ iff $\omega(G+e') > \omega(G)$

András Sebö has proved that the minimal imperfect graphs containeing certain configurations of two $\alpha-$critical edges and one co-critical nonedge are exactly the odd holes or antiholes [8].

S. Markossian, G. Gasparian, I. Karapetian and A. Markosian have also studied such edges and non-edges in conjonction with the Strong Perfect Graph Conjecture [7].

Lemma below proves that the only circular partitionable graphs having $\alpha-$critical edges or co-critical non-edges are webs.

Lemma 2 *Let G(A,B) be a circular partitionable graph. The following statements are equivalent:*

1. $INT(A) = \omega - 1$.
2. $INT(B) = \alpha - 1$.
3. *G(A,B) has an α-critical edge.*
4. *G(A,B) has a cocritical nonedge.*
5. *G(A,B) is a web.*

Proof.

(1) $\Rightarrow$ (5) Let i be an integer such that $|A \cap A + i| = \omega - 1$. Let say that a subset W of A is a *cycle* in A, iff $W = W + i$ and no proper subset of W has this property. Let say that a subset W' of A is a *chain* in A if W' is not a cycle and can be written $\{w\} + i * [0..(r-1)]$ (r is called the *length* of the chain W'). We consider the unique partition of A in cycles and maximal chains.

Fact 1 *There is one and only one chain T in A*

If A doesn't have a chain then $|A \cap (A+i)| = \omega$. Contradiction. If A has two chains then $|A \cap (A+i)| \leq \omega - 2$. Contradiction.

Fact 2 *There is no cycle in A.*

Let u be the element of Z_n such that $(A + B = Z_n \backslash \{u\})$. Let $Z := \{x \in Z_n,\ x = u\ (mod\ i)\}$. Let C be any cycle in A. Note that $|Z| = |C|$.

Since $A + B = Z_n \backslash \{u\}$, there is no $b \in B$ such that $b + C \subset Z$. So, $Z \backslash \{u\}$ is covered only with translated of T. Let r be the length of T. If T is used twice i.e. $\exists b \in B$ such that $b' := b + i * r \in B$ and $b + T \subset Z$, $b' + T \subset Z$, then let $c \in C$: since $|C| \gneq r$, we have $c + (b' - b) = c + ir \in C$, which is in contradiction with the definition of a near-factorization.

Hence Z is covered with only one translated of T and $r = |C| - 1$.

Since B contains at least 2 elements b and b', we have $(b + T) \cap Z = \emptyset$ for instance. Let $t \in T$ and $Z' := \{x \in Z_n,\ x = (b + t)\ (mod(i))\}$. Note that $(b + T) \subset Z'$. Since $|Z'| = |Z| = |T| + 1$, the set $Z' \backslash T$ can't be covered by a cycle C because $|C| \geq 2$ and can't be covered with the help of T because Z' would be covered with 2 pieces T which is impossible as above: contradiction.

Fact 3 *G(A,B) is a web.*

By fact 1 and fact 2, we have $A = T$. Suppose i is not inversible: $Z\backslash\{u\}$ and Z' can't be both covered with the piece A because $|Z'| = |Z\backslash\{u\}| + 1$. So i is inversible, this implies that the graph $G\left(\frac{1}{i}A, \frac{1}{i}B\right)$ is isomorphic to $G(A, B)$ (see [6]. Since $\frac{1}{i}A = \frac{1}{i}T$ is an intervalle, $G(\frac{1}{i}A, \frac{1}{i}B)$ is a web.

(5) $\Rightarrow$ (3) Follows from the definition of a web.

(3) $\Rightarrow$ (2) By definition of a critical edge.

(2) $\Rightarrow$ (5) Since (1) $\Rightarrow$ (5), we have G(B,A) is a web. But this implies that G(A,B) is a web. (see the proof of lemma 2.2 in [1] and note that α is inversible.)

(5) $\Rightarrow$ (4) Follows from the definition of a web.

(4) $\Rightarrow$ (1) By the definition of a cocritical non-edge.

□

2 Intersection degree of of $\omega-$cliques of circular partitionable graph.

2.1 The Parents Lemma

A maximum clique K of G is a *mother* of a vertex $x \in K$ if every maximum clique K' containing x satisfies $|K \cap K'| \geq 2$. Similarly, a maximum stable set S of G is a *father* of a vertex $x \in S$ if every maximum stable set S' containing x satisfies $|S \cap S'| \geq 2$.

Lemma 3 The Parents Lemma *[G. Bacsó, E. Boros, V. Gurvich, F. Maffray, M. Preissmann [1]] If a vertex of a partitionable graph has a father and a mother then the graph has a small transversal.*

Following [1], let an *isoceles triangle* in a circular partitionable graph be three vertices of an $\omega-$clique that can be written as $c,\ c+x,\ c-x$ for some x.

Lemma 4 *Let $G(A, B)$ be a circular partitionable graph such that $A = -A$. If A contains an isoceles triangle then A is a mother of one of its vertices.*

Proof. Let $\{a-i, a, a+i\}$ be an isoceles triangle contained in A. Let $A+j$ be any $\omega-$clique such that $a \in A+j$. Then $j \in -A+a = A+a$, thus $\exists a',\ j = a+a'\ (mod\ n)$.

- if $a = a'$: $(i-a) \in -A = A$, so $((i-a)+j) = i+a \in A+j$. Thus $\{a, i+a\} \subset (A \cap A+j)$.
- if $a \neq a'$, then $\{a, a'\} \subset (A \cap A+j)$.

Hence, A is a mother of a. □

Notes:

1. The condition $A = -A$ is unrestrictive because as pointed out in [1], if $G(A, B)$ is any circular partitionable graph, then there exists $A',\ B'$ such that $A' = -A',\ B' = -B'$ and $G(A', B')$ is isomorphic to $G(A, B)$ (the proof is due to D. De Caen, D.A. Gregory and I.G. Hughes [2]).
2. The preceding lemma is proved in [1] for CGPW graphs (lemma 1.5).

2.2 The strong perfect graph conjecture holds for circular partitionable graphs of degree intersections at least $\frac{\omega}{2}$ and $\frac{\alpha}{2}$.

It is known that every circular partitionable graph with n even has a small transversal (G. Bacsó, E. Boros, V. Gurvich, F. Maffray, M. Preissmann [1]) ; the proof short and elegant is based on this observation:

Lemma 5 *Let $G(A,B)$ a partitionable graph such that $A = -A$ and ω is odd. Then if n is odd, A is a mother of 0, else if n is even, A or $A + \frac{n}{2}$ is a mother of 0.*

Thus, it remains to investigate the case $|A|$ even.

Theorem 1 *If $G(A,B)$ is a circular partitionable graph such that A is even and $INT(A) \geq \frac{\omega}{2}$ then A contains an isoceles triangle or $G(A,B)$ is a CGPW graph.*

Corollary 1 *If $G(A,B)$ is a circular partitionable graph such that $INT(A) \geq \llcorner\frac{\omega}{2}\lrcorner$ and $INT(B) \geq \llcorner\frac{\alpha}{2}\lrcorner$, then $G(A,B)$ has a small transversal or is a CGPW graph, hence is not minimal imperfect unless it is an odd hole or an odd anti-hole.*

2.2.1 Proof of theorem 1

Lemma 6 *A admits an isoceles triangle or A can be partitioned in two subsets A_1 and $A_1 + i$.*

Proof.
By assumption, there is an $\omega-$clique $A+i$ such that $|A \cap A+i| \geq \binom{\omega}{2}$.
Let $A_2 := (A+i) \cap A$ and $A_1 := A \backslash (A+i)$. We have $|A_2| \geq \binom{\omega}{2}$ and $\forall a \in A_2,\ a - i \in A$.
If $\forall a \in A_2,\ a - i \in A_1$ then $|A_1| \geq |A_2|$. So $|A_1| = |A_2| = \binom{\omega}{2}$ and $A = A_1 \cup A_2$ with $A_2 = A_1 + i,\ A_2 \cap A_1 = \emptyset$.
Otherwise, $\exists a \in A_2,\ a - i \notin A_1$ which means $a - i \in A_2$ and $a - 2i \in A$. Hence, $\{a-2i,\ a-i,\ a\}$ is an isoceles triangle in A. □

Lemma 7 *If A is partitioned in two subsets A_1 and $A_1 + i$, then $G(A,B)$ is isomorphic to a CGPW graph.*

Proof.

Fact 1 *i is inversible*

Proof. Follows from *Lemma 2.1 [1].* □

Fact 2 *We may suppose that (A,B) induces a near-factorization in integers.*

Proof. Let $A_1 := \frac{1}{i}A$ and $B_1 := \frac{1}{i}B$. Let u' be such that $A_1 + B_1 = Z_n \backslash \{u'\}$. $\exists a \in A_1,\ b \in B_1$ such that $u' + 1 = a + b$.
Let $A_2 := A_1 - a$ and $B_2 := B_1 - b$.
Then $A_2 + B_2 = Z_n \backslash \{n-1\}$ and $(n-1) \notin A_2,\ 0 \in A_2,\ 0 \in B_2$.
Let A_3 (resp. B_3) be the subset of integers induced by A_2 (resp. B_3), that is $A_3 := \{x \in [0..n], \overline{x} \in A_2\}$ where $\overline{x}$ is the class of x modulo n.
To prove that (A_3, B_3) is a near-factorization in integers we have to show that $max(A_3 + B_3) \leq (n-2)$.

By construction (since $(n-1) \notin A_2$), $A_3 = \cup_{i=1,\ldots,\frac{\omega}{2}}\{a_i,\ a_i+1\}$ with $a_1 = 0$ (the a_i are supposed to be an increasing sequence).

$\forall i = 2, \ldots, \frac{\omega}{2}$, let $d_i := a_i - a_{i-1}$.

Note that since $0 \in A_3$ and $\overline{A_3 + B_3} = A_2 + B_2 = Z_n \backslash \{n-1\}$, all the d_i are even.

Suppose that $max(A_3 + B_3) \geq n$, then $\exists b \in B_3$ and i such that $b + a_i \geq n$. We take the smallest i having this property.

Then d_i is odd: contradiction, hence $max(A_3 + B_3) \leq n - 1$. Since $\{n-1\}$ is not reached, this implies that $max(A_3 + B_3) \leq n - 2$. Thus (A_3, B_3) is a near-factorization in integers. □

Lemma 7 now follows from the theorem given next section. Note that theorem 1 is implied by Lemma 6 and Lemma 7 □

2.2.2 Near-factorizations in integers are CGPW near-factorizations.

Theorem 2 seems to be an already known result ([4], page 10) and the proof is likely to be due to N.G. De Bruijn [5]: I have not been able to get a copy of this article so I have included in this research report my proof of this theorem.

Theorem 2 *Any near-factorization in integers (A, B) is a CGPW near-factorization.*

Proof. *The proof is by induction:*

$P(k)$: 'If (A, B) is a near-factorization in integers such that the prime decomposition of $|A|$ has at most k factors and the prime decomposition of $|B|$ has at most k factors then (A, B) is a CGPW near-factorization.'

Induction Hypothesis (IH): P(1), ..., P(k-1) are true.

Let (A, B) be a near-factorization in integers such that the prime decomposition of $|A|$ has at most k factors and the prime decomposition of $|B|$ has at most k factors.

Case 1 $1 \in A$:

Let ω_1 be the smallest integer such that $[0..\omega_1] \not\subseteq A$. Thus $\omega_1 \geq 2$.

$Q(p)$:'If $\omega_1 \notin A$, ..., $p\omega_1 \notin A$ then $[0...\omega_1 p] \cap B = \omega_1[0...p]$ and $(n-1) \geq (p+1)\omega_1$.'

- *Q(1) is true: if $\exists b \in B \cap [1, \omega_1 - 1]$ then $\{0, b\} \subset (A \cap B)$ which is impossible because A is a clique and B is a stable set. So $B = \{0, b, ...\}$ with $b \geq \omega_1$. There is $A' \in A$ and $b' \in B$ such that $\omega_1 = a' + b'$: $\omega_1 \notin A \Rightarrow b' \neq 0$, thus $b' = \omega_1$ and $B \cap [0..\omega_1] = \{0, \omega_1\}$.*

  Furthermore $(n-1) > max(A + B) \geq \omega_1 + \omega_1 - 1 = 2\omega_1 - 1$.
- *Q(1) true, ..., Q(p-1) true $\Rightarrow$ Q(p) true: if $\exists b \in B \cap [(p-1)\omega_1 + 1, p\omega_1 - 1]$ then $b + 0 = (p-1)\omega_1 + (b - (p-1)\omega_1)$. Contradiction. We have to prove that $p\omega_1 \in B$. There is $a' \in A$ and $b' \in B$ such that $p\omega_1 = a' + b'$: $\omega_1, ..., p\omega_1 \notin A \Rightarrow b' \neq 0, \omega_1, ..., (p-1)\omega_1$, thus $b' = p\omega_1$ and $B \cap [0...p\omega_1] = \omega_1[0...p]$.*

  Furthermore $(n-1) > max(A + B) \geq p\omega_1 + \omega_1 - 1 = (p+1)\omega_1 - 1$.

Hence, we may define α_1 as the least integer such that ($B = \omega_1[0...\alpha_1 - 1]$ or $(\alpha_1\omega_1 \in A)$). Note that $\alpha_1 \geq 2$.

$$A = [0...\omega_1 - 1] \cup A' \ \textit{with} \ A' := A \backslash [0..\omega_1 - 1]$$

$$B = \omega_1[0...\alpha_1 - 1] \cup B' \ \textit{with} \ B' := B \backslash (\omega_1[0..\alpha_1 - 1])$$

If $A' = \emptyset$ then A is an intervalle, thus G(A,B) is a CGPW graph ([1]).

If $B' = \emptyset$ then ω_1 is inversible ([1]), thus G(B,A) is a CGPW graph, hence G(A,B) is a CGPW graph.

It remains to check the case $A' \neq \emptyset$ and $B' \neq \emptyset$.

Lemma 8

$$A = [0...\omega_1 - 1] + \alpha_1\omega_1 A''$$
$$B = \omega_1[0...\alpha_1 - 1] + \alpha_1\omega_1 B''$$

Corollary 2

$$\omega_1|\omega$$
$$\alpha_1|\alpha$$

Proof. *For $0 \leq p \leq \left(\frac{\alpha\omega}{\alpha_1\omega_1} - 1\right)$,*
let R(p):

$$(p\alpha_1\omega_1 + [0..\alpha_1\omega_1 - 1]) \cap A = \emptyset \textit{ or } \{p\alpha_1\omega_1\} + [0..\omega_1 - 1]$$
$$(p\alpha_1\omega_1 + [0..\alpha_1\omega_1 - 1]) \cap B = \emptyset \textit{ or } \{p\alpha_1\omega_1\} + \omega_1[0..\alpha_1 - 1]$$

- *We have seen that R(0) is true.*
- *R(0) is true, ..., R(p-1) is true $\Rightarrow$ R(p) is true:*

if $p\alpha_1\omega_1 \in A$: Let i be in $[1, \omega_1 - 1]$. $\exists a \in A$, $b \in B$ such that $p\alpha_1\omega_1 + i = a + b$. If $b \neq 0$, then $b \geq \omega_1$: $a < (p\alpha_1\omega_1)$ thus by induction hypothesis on R $\exists p' < p$, $a \in (\{p'\alpha_1\omega_1\} + [0..\omega_1 - 1])$ and $(\{p'\alpha_1\omega_1\} + [0..\omega_1 - 1]) \subset A$. Let $a' := p'\alpha_1\omega_1$.

  - *if $a - i \geq a'$: we have $p\alpha_1\omega_1 + 0 = (a - i) + b$: contradiction.*
  - *if $a - i < a'$ then $(a + \omega_1 - i) < a' + \omega_1$ and $(a + \omega_1 - i) > a \geq a'$ so $a' + 1 \leq a + \omega_1 - i \leq a' + \omega_1 - 1$ $(*)$. We have $p\alpha_1\omega_1 + \omega_1 = (a + \omega_1 - i) + b$: so $b = \omega_1$ and $(a + \omega_1 - i) = p\alpha_1\omega_1$ is a multiple of ω_1 contridicting $(*)$.*

  So $b = 0$. It follows $(\{p\alpha_1\omega_1\} + [0..\omega_1 - 1]) \subset A$. Since $\omega_1[0..\alpha_1 - 1] \subset B$ (R(0)), this implies:

$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap A = \{p\alpha_1\omega_1\} + [0..\omega_1 - 1]$$
$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap B = \emptyset$$

if $p\alpha_1\omega_1 \in B$: $\forall 0 \leq i < \alpha_1$, let S(i):

$$((\{p\alpha_1\omega_1\} + i * \omega_1) + [0...\omega_1 - 1]) \cap B = \{p\alpha_1\omega_1 + i * \omega_1\}$$

  - *S(0) is true: Since $[0...\omega_1 - 1] \subset A$, we have $(\{p\alpha_1\omega_1\} + [0...\omega_1 - 1]) \cap B = \{p\alpha_1\omega_1\}$.*
  - *S(0) is true, ..., S(i-1) is true $\Rightarrow$ S(i) is true: If $(p\omega_1\alpha_1 + i * \omega_1) \notin B$ then $p\omega_1\alpha_1 + i * \omega_1 = a' + b'$ with $0 < a', b' < (p\omega_1\alpha_1 + i * \omega_1)$. By induction hypothesis on R and S: b' is a multiple of ω_1, so $a' \in A$ is a multiple of ω_1. By induction hypothesis on R and S again, this means that $a' = j\alpha_1\omega_1$ with $j > 0$. So $b' = ((p - j)\alpha_1\omega_1 + i * \omega_1)$. By induction hypothesis on R, this implies $((p - j)\alpha_1\omega_1) \in B$. Then we have $p\alpha_1\omega_1 + 0 = (j\alpha_1\omega_1) + (p - j)\alpha_1\omega_1$: contradiction because $j\alpha_1\omega_1 = a' \in A$ and $(p - j)\alpha_1\omega_1 \in B$. So $(p\omega_1\alpha_1 + i * \omega_1) \in B$. Since $[0...\omega_1 - 1] \subset A$, $S(i)$ is true.*

Since S(0), ..., S$(\alpha_1 - 1)$ *are true, and* $[0..\omega_1 - 1] \subset A$,

$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap A = \emptyset$$
$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap B = p\alpha_1\omega_1 + \omega_1[0..\alpha_1 - 1]$$

if $p\alpha_1\omega_1 \notin A, \notin B$: $\forall 0 \leq i < \alpha_1$,

Let S'(i);

$$((\{p\alpha_1\omega_1\} + i * \omega_1) + [0...\omega_1 - 1]) \cap A = \emptyset$$
$$((\{p\alpha_1\omega_1\} + i * \omega_1) + [0...\omega_1 - 1]) \cap B = \emptyset$$

- *S'(0) is true : let* $x \in \{p\alpha_1\omega_1\} + [0...\omega_1 - 1]$ *: we have* $p\alpha_1\omega_1 = a + b$ *with* $0 < a, b < p\alpha_1\omega_1$ *and* $a \in A$, $b \in B$. *By induction hypothesis on R,* $b = j\omega_1$ *and* $a = i\omega_1$ *hence* $x + 0 = p\alpha_1\omega_1 + r = (a + r) + j\omega_1$. *Since* $(a + r) \in A$ *and* $j \neq 0$, *we have* $x \notin A$ *and* $x \notin B$.
- *S'(0) is true, ..., S'(i-1) is true* $\Rightarrow$ *S'(i) is true : apply the induction hypothesis on S' and then the demonstration for S'(0).*

Since S'(0), ..., S'$(\alpha_1 - 1)$ *are true,*

$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap A = \emptyset$$
$$(\{p\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap B = \emptyset$$

Let j *be the greatest integer such that* $j\alpha_1\omega_1 < \alpha\omega = n - 1$.

To prove Lemma 8, it remains to prove that :

$$A = \cup_{i=0}^{j-1}((\{i\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap A)$$
$$B = \cup_{i=0}^{j-1}((\{i\alpha_1\omega_1\} + [0..\alpha_1\omega_1 - 1]) \cap B)$$

- *claim* $j\alpha_1\omega_1 \notin B$ *: otherwise since* $A' \neq \emptyset$, *we have* $\alpha_1\omega_1 \in A$ *and so* $max(A + B) \geq (\alpha_1\omega_1 + j\alpha_1\omega_1) = (j + 1)\alpha_1\omega_1 > \alpha\omega - 1$.
- *claim* $j\alpha_1\omega_1 \notin A$ *: otherwise since* $B' \neq \emptyset$, *there is* $b > \alpha_1\omega_1 \in B$. *So if* $j\alpha_1\omega_1 \in A$ *then* $max(A + B) > (j + 1)\alpha_1\omega_1 > \alpha\omega - 1$.

Since $j\alpha_1\omega_1 \notin A, \notin B$, *it follows* $\forall x \in ((A + B) \cap (j\alpha_1\omega_1 + [0..\alpha_1\omega_1 - 1])$ *that* $x \notin A$ *and* $x \notin B$ *(see the case* $p\alpha_1\omega_1 \notin A, \notin B$*). Thus, we are done.*

□

- *P(1) is true : since* $\omega_1 | \omega$, *we have* $\omega = \omega_1$. *Obviously, also* $\alpha = \alpha_1$. *So* $A' = \emptyset$ *and* $B' = \emptyset$. *Thus (A,B) is a CGPW near-factorization.*
- *P(1) is true, ... ,P(k-1) is true* $\Rightarrow$ *P(k) is true :*

Lemma 9

$$A'' + B'' = \left[0..\frac{\alpha\omega}{\alpha_1\omega_1} - 1\right]$$

Proof. *Let* $x \in [0..\frac{\alpha\omega}{\alpha_1\omega_1} - 1]$. *Then* $\alpha_1\omega_1 x \in [0..\alpha\omega - 1]$. *So* $\exists a \in A$, $b \in B$ *such that* $\alpha_1\omega_1 x = a + b$.

Hence, $\exists a' \in A''$, $0 \leq a_1 \leq \omega_1 - 1$, *such that* $a = a_1 + \alpha_1\omega_1 a'$ *and* $\exists b' \in B''$, $0 \leq b_1 \leq \alpha_1 - 1$, *such that* $b = \omega_1 b_1 + \alpha_1\omega_1 b'$.

Thus $\alpha_1\omega_1(x - a' - b') = a_1 + \omega_1 b_1 \leq \omega_1\alpha_1 - 1$.

Hence $x = a' + b'$ *which means that* $x \in A'' + B''$.

To conclude, note that $|A''| * |B''| = \frac{\alpha\omega}{\alpha_1\omega_1}$. □

Thus we may use the Induction Hypothesis for (A",B") : G(A",B") is a CGPW graph $C[\omega_2, \alpha_2, ..., \omega_p, \alpha_p]$.

Then G(A,B) is the CGPW graph $C[\omega_1, \alpha_1, \omega_2, \alpha_2, ..., \omega_p, \alpha_p]$. *Thus P(k) is true.*

Case 2 $1 \notin A$ *:*

Apply case 1 to (B,A) : G(B,A) is a CGPW graph, hence G(A,B) is a CGPW graph.

□

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