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Abstract: We call a T-vertex of a graph $G = (V, E)$ a vertex z whose neighbourhood $N(z)$ in G induces a triangulated graph, and we show that every C_4-free Berge graph either is a clique or contains at least two non-adjacent T-vertices. An easy consequence of this result is that every C_4-free Berge graph admits a T-elimination scheme, *i.e.* an ordering $[v_1, v_2, \ldots, v_n]$ of its vertices such that v_i is a T-vertex in the subgraph induced by $v_i, \ldots, v_n$ in G.

Keywords: lexicographic breadth-first search, triangulated graph, perfectness

1 Introduction

A graph is called *perfect* if, for each of its induced subgraphs, the chromatic number equals the clique number. Since Berge [1] introduced perfect graphs, a lot of partial results have been obtained on the Strong Perfect Graph Conjecture (SPGC) which claims that a graph is perfect if and only if it is a Berge graph, that is, it contains no chordless cycle on odd number of vertices, and no complement of such a cycle. Among the particular classes of graphs which have been shown to satisfy the SPGC we can find every class of H-free graphs, for each graph H on four vertices but C_4 (the chordless cycle on four vertices) and its complement $2K_2$. It can be noticed that these two exceptions are in fact only one exception, since a graph is perfect if and only if its complement is (see [3]).

The perfection of C_4-free Berge graphs cannot be approached without a good knowledge of the structure of these graphs, and this knowledge is lacking. Our aim in this paper is to show that a special vertex exists in every C_4-free Berge graph; the immediate consequence of this is that C_4-free Berge graphs (which form an hereditary family) admit elimination schemes (obtained by considering a special vertex, removing it from the graph and repeating this operation on the remaining graph until an empty graph is found).

For us, a special vertex will be a *T-vertex*, that is, a vertex z whose neighbourhood is triangulated (it contains no chordless cycle on four vertices or more). To find it, we will use the lexicographic breadth-first search algorithm [4] (abbreviated LexBFS), which orders the vertices of the graph such that the last vertex in the ordering is the T-vertex we are looking for.

2 Graph decomposition using LexBFS

The algorithm LexBFS has been proposed by Rose, Tarjan and Leuker [4] in order to give a linear recognition algorithm for triangulated graphs. It turned out that for triangulated graphs, the last vertex in the order found by LexBFS is always a *simplicial* vertex, i.e. its neighbourhood induces a clique. Moreover, it has been shown [5] that for i-triangulated graphs (that is, graphs where the vertices of every odd cycle induce a triangulated graph) the last vertex in the order given by LexBFS is always a multi-partite graph.

The description of LexBFS that we present here is due to Golumbic [2]. We attach to every vertex a label which consists of a set of numbers listed in decreasing order. The labels are compared using lexicographic order. Then the algorithm is the following:

Algorithm LexBFS

Input: a connected graph $G = (V, E)$ with n vertices
Output: a function $\sigma : \{1, 2, \ldots, n\} \to V$ (an order on V)

assign the label $\emptyset$ to each vertex;
for $i := n$ down to 1 do
  pick an unnumbered vertex v with a largest label (in lexicographic order);
  $\sigma(i) := v$; {comment: this assigns to v the number i}
  for each unnumbered vertex $t \in N(v)$ do
    add i to the label of t (at the end)

Notice that in the order σ obtained by this algorithm, the *last* vertex which has been chosen in the graph is $\sigma(1)$ (that we denote z all along the paper). This is the special vertex both in the case of triangulated graphs, and in the case of i-triangulated graphs. And this will be the special vertex in the case of C_4-free Berge graphs too.

As it can be easily seen, LexBFS can be applied to every graph. Based on the order given by LexBFS, a graph decomposition and structural results can be found, which are also valid for every graph. We give them below, in this section. In the next section, we will study the particular properties of C_4-free Berge graphs with regard to this decomposition. In Section 4, we will prove the existence of at least two non-adjacent T-vertices in every C_4-free Berge graph different from a clique.

Let $G = (V, E)$ be a graph with at least two vertices. Define a partition $H_0, H_1, \ldots, H_p, H'_p$ of V as follows. Put $H_0 = \{\sigma(n)\}$ and notice that $H_0 \subseteq V - \{z\}$ (recall the notation $z = \sigma(1)$). By induction, assume that $H_i \subseteq V - \{z\}$ is already defined (for some $i \geq 0$) and set

$H'_i = \{h' \in V \mid \text{for all}\, h \in H_i, \sigma^{-1}(h) > \sigma^{-1}(h')\}$.

Furthermore, partition H_i into three subsets:

$T_i = H_i \cap N(z)$
$U_i = \{h \in H_i - T_i \mid N_{H'_i}(h) \neq \emptyset\}$, and
$V_i = H_i - (T_i \cup U_i)$.

Now, define $H_{i+1} = \cup_{u \in U_i} N(u) \cap H'_i$ (this last set will be denoted $N_{H'_i}(U_i)$). Assume this construction is repeated p times, so that we have the sets $H_0, H_1, \ldots, H_p$ (which are all non-empty) and H'_p. Let us say that a set $R \subseteq V$ is *compact* if every $u \in V - R$ either satisfies $\sigma^{-1}(r) > \sigma^{-1}(u)$ for any $r \in R$, or satisfies $\sigma^{-1}(r) < \sigma^{-1}(u)$ for any $r \in R$. In [5], two of the authors proved that (see Fig. 1.):

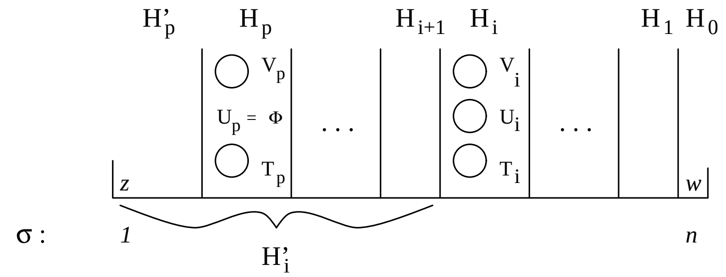

Fig. 1. The sets H_i, H'_i $(i = 0, 1, \ldots, p)$

Claim 1 *For every graph $G = (V, E)$ we have:*
(P1) for all i $(0 \leq i \leq p-1)$, the set H_{i+1} is compact and $H_{i+1} \subseteq H'_i$ (so that $H'_{i+1} \subset H'_i$);
(P2) for all i $(0 \leq i \leq p-1)$, the set $H_i \cup H_{i+1}$ is compact (i.e. no $v \in V$ satisfies $\sigma^{-1}(v) > \sigma^{-1}(h')$ for every $h' \in H_{i+1}$, and $\sigma^{-1}(v) < \sigma^{-1}(h)$ for every $h \in H_i$);
(P3) every $h' \in H'_{i+1}$ has $N(h') \subseteq H'_{i+1} \cup H_{i+1} \cup T_i \cup T_{i-1} \cup \ldots \cup T_0$ $(0 \leq i \leq p-1)$.

An immediate consequence of (P1) and (P2) is that $H_0 \cup H_1 \cup \ldots \cup H_p \cup H'_p$ is a partition of V. Now, since $H_0 \cup H_1 \cup \ldots \cup H_p$ grows when p grows, after a finite number of steps we will have $H_{p+1} = \emptyset$, i.e. $U_p = \emptyset$ (notice that H'_p is never empty, since $z \in H'_p$). This is the value of p we will consider in the rest of the paper.

We will denote the starting vertex $w = \sigma(n)$ (and we maintain the notation $z = \sigma(1)$).

Now, calling a *module* of G an induced subgraph H of G such that any vertex in $G - H$ is either adjacent to all H or to no vertex in H, we prove that:

Claim 2 *For every graph $G = (V, E)$ and every starting vertex w, the partition $H_0 \cup H_1 \cup \ldots \cup H_p \cup H'_p$ has the properties:*
(P4) every vertex in T_i is adjacent to every vertex of H'_{i+1} $(0 \leq i \leq p-1)$;
(P5) every vertex in T_p is adjacent to every vertex of H'_p;
(P6) H'_p is a module of G, and its neighbourhood in $G - H'_p$ is $T_0 \cup T_1 \cup \ldots \cup T_p$.

Proof. If (P4) is not true, then take a vertex $y \in H'_{i+1}$ non-adjacent to some $t \in T_i$ and recall that, during the execution of LexBFS, y has been chosen before z; that is, the label of y was larger (in lexicographic order) than the label of z at the precise moment when y was chosen. Now, by (P3), $N(y) \subseteq H'_{i+1} \cup H_{i+1} \cup T_i \cup T_{i-1} \cup \ldots \cup T_0$ which implies $N(y) - (H'_{i+1} \cup H_{i+1}) \subseteq T_i \cup T_{i-1} \cup \ldots \cup T_0 = N(z) - (H'_{i+1} \cup H_{i+1})$ and the inclusion is strict because of t. But then the label of y was smaller (in lexicographic order) than the label of z at the precise moment when y was chosen, because of t. So z should have been chosen instead of y, a contradiction.

If (P5) is not true, then take $y \in H'_p$ and notice that, since $U_p = \emptyset$, $N(y) \subseteq H'_p \cup T_p \cup T_{p-1} \cup \ldots \cup T_0$. As before, $N(y) - H'_p \subset T_p \cup T_{p-1} \cup \ldots \cup T_0 = N(z) - H'_p$, so z should have been chosen before y by LexBFS, a contradiction.

Statement (P6) is now easy to deduce: every vertex x in $G - H'_p$ is either in some T_i $(0 \leq i \leq p)$ or in no such set. In the first case, x is adjacent to all H'_p by (P4) and (P5). In the second one, $x \in (H_0 - T_0) \cup (H_1 - T_1) \cup \ldots \cup (H_{p-1} - T_{p-1}) \cup V_p$ (since $U_p = \emptyset$) and by (P3) x has no neighbour in H'_p.∎

Finally, we can add these two very useful properties:

Claim 3 *For every graph $G = (V, E)$ we have:*
(P7) if x and y are distinct vertices in H_i, there exists a chordless path joining x, y whose internal vertices are in $H_0 \cup H_1 \cup \ldots \cup H_{i-1}$ $(1 \leq i \leq p)$;
(P8) if G is a Berge graph and x, y are distinct vertices in H_i, then all the chordless paths of length at least three joining x, y in $G - xy$ (i.e. the graph obtained by removing, if it exists, the edge xy) and whose internal vertices are in $H_{i+1} \cup \ldots \cup H_p \cup H'_p$ have the same parity $(1 \leq i \leq p)$.

Proof. Statement (P7) is easy to prove by induction. Obviously, if $x, y \in H_1$ then either they are adjacent (and we have a path with only one edge and no internal vertices) or they are not adjacent and then the path of length 2 given by x, w, y is the one we need. Now, if we assume this is true for $i-1$, then, as before, only the case where $xy \notin E$ is interesting. Each of $x, y \in H_i$ has a neighbour x_1, respectively y_1 in H_{i-1}. If $x_1 y \in E$ or $y_1 x \in E$, then we have a chordless path of length 2 joining x, y; otherwise, we apply induction hypothesis for $x_1, y_1 \in H_{i-1}$ to find a path joining them, and then add x, y to this path.

To deduce statement (P8), it is sufficient to notice that if two paths with the indicated property and different parities exist, then two chordless cycles with different parities and cardinality at least four may be found by putting together each of these two paths and the path given by (P7).∎

Remark 1 Obviously, if in (P8) the vertices x, y are non-adjacent, then the condition on the length of the paths (at least three) can be dropped.

In the next sections, we investigate C_4-free Berge graphs using these general properties of the decomposition given by LexBFS.

3 Preliminary results

Let G be a connected C_4-free Berge graph and assume we applied to G the decomposition above. The entire notation (for w and z included) is preserved.

In the aim to prove that z is a T-vertex, we notice that $N(z) - H'_p = T_0 \cup \ldots \cup T_p$ and recall that H'_p is a module. Our approach is the following one: in this section, we prove particular properties of the sets T_i $(0 \leq i \leq p)$, with regard to the P_4 (chordless paths on four vertices). Then, in the next section, we prove that z is a T-vertex.

An important role is played in our reasoning (see next section) by the chordless paths on four vertices. We show below that the distribution of their vertices in the non-empty sets T_i obeys to some constraints.

For that, we need some supplementary notation. In the proofs below, we will define sets A_1, B_1, C_1, D_1 which are always non-empty. The notation a_1 (respectively b_1, c_1, d_1) will then concern an arbitrary (but fixed) vertex in A_1 (respectively B_1, C_1, D_1). The notation $(ab)_1$ will concern an arbitrary (but fixed) vertex in $A_1 \cap B_1$, whenever this set is (or is supposed) non-empty (and similarly for the other intersections). The notation $(a.b)_1$ will concern an arbitrary (but fixed) vertex in $A_1 - B_1$, whenever this set is (or is supposed) non-empty (and similarly for the other set differences).

A path will be deboted by $[v_1, v_2, \ldots, v_k]$ and a cycle by $[v_1, v_2, \ldots, v_k, v_1]$. We use the classical notation P_k (respectively C_k) to denote a chordless path (respectively cycle) on $k \geq 4$ vertices.

Lemma 1 *For every i $(0 \leq i \leq p)$, T_i is P_4-free.*

Proof. Assuming the contrary, let T_i be a set which contains a P_4 denoted $[a, b, c, d]$. Since T_0 contains at most one vertex, we have $i \geq 1$. Define
$A_1 = N_{U_{i-1}}(a)$,
$B_1 = N_{U_{i-1}}(b)$,
$C_1 = N_{U_{i-1}}(c)$ and
$D_1 = N_{U_{i-1}}(d)$.
These sets are non empty since $H_i = N_{H'_{i-1}}(U_{i-1})$ (by definition). We have that:
- $A_1 \cap C_1 = \emptyset$; otherwise $[a, (ac)_1, c, z, a]$ is a C_4.
- $A_1 \cap D_1 = \emptyset$; otherwise $[a, (ad)_1, d, z, a]$ is a C_4.
- $B_1 \cap D_1 = \emptyset$; otherwise $[b, (bd)_1, d, z, b]$ is a C_4.
- $A_1 \subseteq B_1$ or $D_1 \subseteq C_1$; otherwise the paths $[(a.b)_1, a, z, d, (d.c)_1]$ and $[(a.b)_1, a, b, c, d, (d.c)_1]$ contradict (P8).
- $A_1 \subseteq B_1$ can be assumed without loss of generality, because of the preceding affirmation (since edges ab, cd are symmetrical with respect to the P_4 $[a, b, c, d]$).
- $C_1 \subseteq B_1$, otherwise $[a_1, a, z, c, (c.b)_1]$ and $[a_1, b, c, (c.b)_1]$ are paths which contradict (P8).
- $C_1 \cap D_1 = \emptyset$ because of $C_1 \subseteq B_1$ and $B_1 \cap D_1 = \emptyset$.

Then the chordless paths $[c_1, c, d, d_1]$ and $[c_1, b, z, d, d_1]$ contradict (P8).∎

Lemma 2 *For every i ($0 \leq i \leq p-1$), a P_4 $[a,b,c,d]$ with $a,b,c,d \in T_i \cup T_{i+1}$ satisfies none of the properties below:*
i) $a,b,c \in T_i$ and $d \in T_{i+1}$;
ii) $a \in T_i$ and $b,c,d \in T_{i+1}$;
iii) $a,c \in T_i$ and $b,d \in T_{i+1}$;
iv) $a,d \in T_i$ and $b,c \in T_{i+1}$;
v) $b,c \in T_i$ and $a,d \in T_{i+1}$.

Proof. By contradiction, we assume that a P_4 satisfying i), ii), iii), iv) or v) exists, and we consider each case separately in the aim to obtain a contradiction. It is not difficult to see that in each case we have $i \geq 1$. As a general observation, recall that no vertex in H_{i+1} has neighbours in $H_{i-1} - T_{i-1}$ (because of (P3)). This observation will be often used in the sequel without any specification.

Case i) : $a,b,c \in T_i$ and $d \in T_{i+1}$.
Because of $H_{i+1} = N_{H'_i}(U_i)$, there exists $d' \in U_i$ such that $d'd \in E$. Denote
$A_1 = N_{U_{i-1}}(a)$,
$B_1 = N_{U_{i-1}}(b)$,
$C_1 = N_{U_{i-1}}(c)$ and
$D_1 = N_{U_{i-1}}(d')$.
Because of $H_i = N_{H'_{i-1}}(U_{i-1})$, these sets are non-empty. We have:

- $d'b \notin E$; otherwise $[b,d',d,z,b]$ is a C_4.
- $d'a \notin E$; otherwise $[a,d',d,z,a]$ is a C_4.
- $A_1 \cap C_1 = \emptyset$; otherwise $[a,(ac)_1,c,z,a]$ is a C_4.
- $A_1 \cap D_1 = \emptyset$; otherwise $[a,(ad)_1,d',d,z,a]$ is a C_5.
- $B_1 \cap D_1 = \emptyset$; otherwise $[b,(bd)_1,d',d,z,b]$ is a C_5.
- $d'c \in E$, since in the contrary case we successively have: $D_1 \cap C_1 = \emptyset$ (otherwise $[c,(cd)_1,d',d,c]$ is a C_4), $A_1 \subseteq B_1$ (otherwise the paths $[(a.b)_1,a,z,d,d',d_1]$ and $[(a.b)_1,a,b,c,d,d',d_1]$ contradict (P8)), $C_1 \subseteq B_1$ (otherwise the paths $[a_1,a,z,c,(c.b)_1]$ and $[a_1,b,c,(c.b)_1]$) contradict (P8)) and consequently the paths $[c_1,b,z,d,d',d_1]$ and $[c_1,c,d,d',d_1]$ give again a contradiction with (P8).
- $D_1 \subseteq C_1$, since in the contrary case we successively have: $B_1 \subseteq C_1$ (otherwise the paths $[(d.c)_1,d',c,b,(b.c)_1]$ and $[(d.c)_1,d',d,z,b,(b.c)_1]$ contradict (P8)), $A_1 \cap B_1 = \emptyset$ (because of $B_1 \subseteq C_1$ and $A_1 \cap C_1 = \emptyset$) and consequently the paths $[a_1,a,b,b_1]$ and $[a_1,a,z,c,b_1]$ contradict (P8).
- $A_1 \subseteq B_1$; otherwise the paths $[(a.b)_1,a,b,c,d_1]$ and $[(a.b)_1,a,z,d,d',d_1]$ contradict (P8).

Then the chordless paths $[a_1,b,c,d_1]$ and $[a_1,a,z,c,d_1]$ contradict (P8).

Case ii) : $a \in T_i$ and $b,c,d \in T_{i+1}$.
Because of $H_{i+1} = N_{H'_i}(U_i)$, there exist $b',d' \in U_i$ such that $b'b, d'd \in E$ (so $b' \neq d'$ otherwise a C_4 is induced with b,z,d). Define
$A_1 = N_{U_{i-1}}(a)$,
$B_1 = N_{U_{i-1}}(b')$ and
$D_1 = N_{U_{i-1}}(d')$.
As usual, these sets are non-empty. We have:

- $d'b \notin E$; otherwise $[d',b,z,d,d']$ is a C_4.
- $d'c \in E$; otherwise the paths $[d',d,z,a]$ and $[d',d,c,b,a]$ contradict (P8).
- $b'd \notin E$; otherwise $[b,b',d,z,b]$ is a C_4.
- $b'd' \notin E$; otherwise $[b,b',d',d,z,b]$ is a C_5.
- $A_1 \cap D_1 = \emptyset$; otherwise $[d',(ad)_1,a,b,c,d']$ is a C_5.
- $B_1 \cap D_1 = \emptyset$; otherwise either $[d',(bd)_1,b',c,d']$ is a C_4 (in the case where $b'c \in E$) or $[d',(bd)_1,b',b,c,d']$ is a C_5 (in the case where $b'c \notin E$).
- $b'c \in E$; otherwise the paths $[b_1,b',b,c,d',d_1]$ and $[b_1,b',b,z,d,d',d_1]$ contradict (P8).
- $A_1 \cap B_1 = \emptyset$; otherwise the paths $[(ab)_1,b',c,d',d_1]$ and $[(ab)_1,a,z,d,d',d_1]$ contradict (P8).

• $ab' \in E$; otherwise the paths $[a_1, a, b, b', b_1]$ and $[a_1, a, z, c, b', b_1]$) contradict (P8).
Then $[b', a, z, c, b']$ is a C_4, a contradiction.

Case iii) : $a, c \in T_i$ and $b, d \in T_{i+1}$.

Because of $H_{i+1} = N_{H_i'}(U_i)$, there exist $b', d' \in U_i$ such that $b'b, d'd \in E$ (so $b' \neq d'$ otherwise a C_4 is induced with b, z, d). Define

$A_1 = N_{U_{i-1}}(a)$,
$B_1 = N_{U_{i-1}}(b')$,
$C_1 = N_{U_{i-1}}(c)$ and
$D_1 = N_{U_{i-1}}(d')$.

As usual, these sets are non-empty. Then we have:

• $d'a \notin E$; otherwise $[d', a, z, d, d']$ is a C_4.
• $d'b \notin E$; otherwise $[d', b, z, d, d']$ is a C_4.
• $b'd \notin E$; otherwise $[b', d, z, b, b']$ is a C_4.
• $b'd' \notin E$; otherwise $[b', d', d, z, b, b']$ is a C_5.
• $A_1 \cap C_1 = \emptyset$; otherwise $[c, (ac)_1, a, z, c]$ is a C_4.
• $A_1 \cap D_1 = \emptyset$; otherwise $[d', (ad)_1, a, z, d, d']$ is a C_5.
• $d'c \in E$, since in the contrary case we have $D_1 \cap C_1 = \emptyset$ (otherwise $[c, (dc)_1, d', d, c]$ is a C_4) and then the paths $[d_1, d', d, c, b, a, a_1]$ and $[d_1, d', d, z, a, a_1]$ contradict (P8).
• $C_1 \cap D_1 = \emptyset$; otherwise, $[(cd)_1, c, b, a, a_1]$ and $[(cd)_1, d', d, z, a, a_1]$ contradict (P8).
• $b'c \in E$, since in the contrary case we have $B_1 \cap C_1 = \emptyset$ (otherwise $[c, (bc)_1, b', b, c]$ is a C_4) and then the paths $[b_1, b', b, c, d', d_1]$ and $[b_1, b', b, z, d, d', d_1]$ contradict (P8).
• $b'a \notin E$; otherwise $[b', a, z, c, b']$ is a C_4.
• $A_1 \cap B_1 = \emptyset$; otherwise $[a, (ab)_1, b', b, a]$ is a C_4.
• $B_1 \cap D_1 = \emptyset$; otherwise the paths $[(bd)_1, b', b, a, a_1]$ and $[(bd)_1, d', d, z, a, a_1]$ contradict (P8).
• $B_1 \cap C_1 = \emptyset$; otherwise the paths $[(bc)_1, c, d', d_1]$ and $[(bc)_1, b', b, z, d, d', d_1]$ contradict (P8).

Then the paths $[a_1, a, b, b', b_1]$ and $[a_1, a, z, c, b', b_1]$ contradict (P8).

Case iv) : $a, d \in T_i$ and $b, c \in T_{i+1}$.

Denote $A_1 = N_{U_{i-1}}(a)$ and $D_1 = N_{U_{i-1}}(d)$ (non-empty sets of vertices). Then $A_1 \cap D_1 = \emptyset$ (otherwise $[d, (ad)_1, a, b, c, d]$ is a C_5) and the paths $[a_1, a, b, c, d, d_1]$ and $[a_1, a, z, d, d_1]$ contradict (P8).

Case v) : $b, c \in T_i$ and $a, d \in T_{i+1}$.

Because of $H_{i+1} = N_{H_i'}(U_i)$, there exist $a', d' \in U_i$ such that $a'a, d'd \in E$ (so $a' \neq d'$ otherwise a C_4 is induced with a, z, d). Define

$A_1 = N_{U_{i-1}}(a')$,
$B_1 = N_{U_{i-1}}(b)$,
$C_1 = N_{U_{i-1}}(c)$ and
$D_1 = N_{U_{i-1}}(d')$.

As usual, these sets are non-empty and we have:

• $a'd \notin E$; otherwise $[a', a, z, d, a']$ is a C_4.
• $d'a \notin E$; otherwise $[d', a, z, d, d']$ is a C_4.
• $a'd' \notin E$; otherwise $[a', a, z, d, d', a']$ is a C_5.
• $a'c \notin E$; otherwise $[a', a, z, c, a']$ is a C_4.
• $d'b \notin E$; otherwise $[d', d, z, b, d']$ is a C_4.
• $A_1 \cap C_1 = \emptyset$; otherwise $[c, (ac)_1, a', a, z, c]$ is a C_5.
• $B_1 \cap D_1 = \emptyset$; otherwise $[(bd)_1, d', d, z, b, (bd)_1]$) is a C_5.
• $a'b \in E$ or $d'c \in E$, since in the contrary case we successively have: $A_1 \cap B_1 = \emptyset$ (otherwise $[b, (ab)_1, a', a, b]$ is a C_4), $C_1 \cap D_1 = \emptyset$ (otherwise $[c, (cd)_1, d', d, c]$ is a C_4) and then the paths $[a_1, a', a, z, d, d', d_1]$ and $[a_1, a', a, b, c, d, d', d_1]$ contradict (P8).

• $d'c \in E$ can be assumed without loss of generality, because of the preceding affirmation (since edges ab, cd are symmetrical with respect to the P_4 $[a, b, c, d]$).

• $a'b \notin E$, since in the contrary case we successively have: $a_1 b \in E$ or $d_1 c \in E$ (otherwise the paths $[a_1, a', b, c, d', d_1]$ and $[a_1, a', a, z, d, d', d_1]$ contradict (P8)), and we assume without loss of generality (for symmetry reasons) that $a_1 b \in E$; $d_1 c \notin E$ (otherwise the paths $[a_1, b, c, d_1]$ and $[a_1, a', a, z, d', d, d_1]$ contradict (P8)) and then the paths $[a_1, b, c, d', d_1]$ and $[a_1, b, z, d, d', d_1]$ contradict (P8).

• $A_1 \cap B_1 = \emptyset$; otherwise $[b, (ab)_1, a', a, b]$ is a C_4.

• $D_1 \cap C_1 = \emptyset$ since in the contrary case we have $(cd)_1 a' \notin E$ (otherwise $[a', (cd)_1, c, z, a, a']$ is a C_5) and then the paths $[(cd)_1, d', d, z, a, a', a_1]$ and $[(cd)_1, c, b, a, a', a_1]$ contradict (P8).

• $b_1 c \in E$; otherwise the paths $[b_1, b, c, d', d_1]$ and $[b_1, b, z, d, d', d_1]$ contradict (P8).

Then the paths $[b_1, c, z, a, a', a_1]$ and $[b_1, b, a, a', a_1]$ contradict (P8).∎

4 Main result

This section is devoted to the result we announced, that we state here:

Theorem 1 *The vertex $z = \sigma(1)$ is a T-vertex.*

Proof. We use induction on the number of vertices of the graph. The theorem is obviously true for a graph with two vertices (recall that we assumed G is connected). Now, assume the theorem is true for all connected graphs with strictly less than n vertices, and let us prove it for $G = (V, E)$ which has n vertices. Recall that H'_p is a module of G, whose neighbourhood in $G - H'_p$ is $T_0 \cup T_1 \cup \ldots \cup T_p$.

Claim 4 *The theorem is true if H'_p has at least two vertices.*

Proof. Consider the precise moment where the first vertex in H'_p (i.e. the vertex y with $\sigma^{-1}(y) = max\{\sigma^{-1}(x) \mid x \in H'_p\}$) has been chosen by the algorithm LexBFS. As H'_p is a module, and $T_0 \cup \ldots \cup T_p$ contains only vertices that have been numbered *before* every vertex in H'_p, all the vertices in H'_p have the same label as y at the indicated moment. Therefore, the execution of LexBFS restricted to the subgraph $G[H'_p]$ (induced by H'_p in G) is exactly the one we would have if $G[H'_p]$ was considered as a separate, entire graph and LexBFS was applied to it. Then, we can apply the induction hypothesis to deduce that the neighbourhood of z in $G[H'_p]$ is triangulated.

We consider $V' = V - H'_p \cup \{z\}$ and observe that the execution of LexBFS on G restricted to the subgraph $G[V']$ is identical to the one we would have if $G[V']$ was considered as a separate, entire graph and LexBFS was applied to it (since the vertices of $H'_p - \{z\}$ are chosen at the end of the execution on G, so they do not influence the numbering of the vertices in $G - H'_p$). Moreover, since H'_p has at least two vertices, V' has at most $n-1$ vertices and we can apply the induction hypothesis for $G[V']$ to deduce that $T_0 \cup \ldots \cup T_p$ (i.e. $N(z) - H'_p$) is triangulated. As $H'_p \cap N(z)$ is triangulated (see above) and $T_0 \cup \ldots \cup T_p$ is triangulated too, $N(z)$ (which is the complete join of these two sets) is triangulated too. In fact, we can notice that at least one of the two sets is a clique, otherwise a C_4 can be found by taking a pair of non-adjacent vertices in each set.∎

Claim 5 *The theorem is true if $H'_p = \{z\}$.*

Proof. Suppose by contradiction that z is not a T-vertex. Then there exists a chordless cycle $C = [v_1, v_2, ..., v_k, v_1]$ in $N(z)$ with at least four vertices. Moreover, the fact that G is a C_4-free Berge graph implies that C contains at least six vertices. Now, call i the minimum index such that C has at least one vertex in T_i, and assume without loss of generality that $v_1 \in T_i$. So, for every $v \in C$, we have $v \in T_j$ with $j \geq i$.

By $(P4)$, if $t_j \in T_j, t_k \in T_k$ with $|j-k| \geq 2$, then $t_j t_k \in E$. Now, since v_3, v_4 and v_5 are non-adjacent to v_1, these three vertices belong necessarily to $T_i \cup T_{i+1}$. The vertex v_2 is in $T_i \cup T_{i+1} \cup T_{i+2}$, since it is non-adjacent to v_4 and v_5. Now we distinguish three cases :

1. v_2 belongs to T_{i+2}.

   In this case, v_4 and v_5 must be in T_{i+1} (in the contrary case they are adjacent to v_2). Thus, $v_3 \in T_i$, otherwise the P_4 $[v_2, v_3, v_4, v_5]$ verifies the property i) in Lemma 2, a contradiction. Now, v_6 is not adjacent to v_3 and v_2, and must be (by (P4)) in T_{i+1}. But $[v_3, v_4, v_5, v_6]$ is a P_4 which verifies the property ii) in Lemma 2, a contradiction.

2. v_2 belongs to T_{i+1}.

   Since the P_4 $[v_1, v_2, v_3, v_4]$ cannot verify the property ii), iii) or iv) in Lemma 2, the two vertices v_3 and v_4 must be in T_i. If $v_5 \in T_i$ (resp. $v_5 \in T_{i+1}$) then the P_4 $[v_2, v_3, v_4, v_5]$ verifies the property ii) (respectively the property v)) in Lemma 2, a contradiction.

3. v_2 belongs to T_i.

   One of v_3 or v_4 belongs to T_{i+1}, otherwise $[v_1, v_2, v_3, v_4]$ is a P_4 of T_i, a contradiction with Lemma 1. As $[v_1, v_2, v_3, v_4]$ cannot verify the property i) in Lemma 2, v_3 must be in T_{i+1}. Now, we can perform a change of notation as follows: for every $(1 < i \leq p$, $v'_{i-1} = v_i$; moreover, $v'_k = v_1$. Then for v'_1 and v'_2 we have the situation in case 2 (with v'_1 instead of v_1 and v'_2 instead of v_2), so we can conclude as in case 2.∎

Claim 4 and Claim 5 prove the theorem.∎

Using Theorem 1, there is no difficulty to see that the order $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ given by LexBFS has the following property: for every $j \in \{1, 2, \ldots, n\}$, the vertex $\sigma(j)$ is a T-vertex in the graph $G[\{\sigma_j, \sigma_{j+1}, \ldots, \sigma(n)]$. This scheme is called a *T-elimination scheme* and it gives a regard on the structure of C_4-free Berge graphs which is similar to the one we have for triangulated and i-triangulated graphs.

Theorem 1 has the following easy corollary:

Corollary 1 *Every C_4-free Berge graph G which is not a clique has at least two non-adjacent T-vertices.*

Proof. Since G is not a clique, there exists at least one non-universal vertex w in G. Perform LexBFS by choosing firstly the vertex w (so that $\sigma(n) = w$). Then $z = \sigma(1)$ is a T-vertex, by Theorem 1. Moreover, z is not universal, since $zw \notin E$. Perform again LexBFS, this time by choosing z as the first vertex (i.e. $\sigma'(n) = z$) and denote $x = \sigma'(1)$. Then z, x are two non-adjacent T-vertices. ∎

References

[1] C. Berge - Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, Wiss. Z. Martin Luther Univ., Halle Wittenberg (1961), 114–115.

[2] M. C. Golumbic, "Algorithmic graph theory and perfect graphs", Academic Press (1980).

[3] L. Lovász - A characterization of perfect graphs, *J. Combin. Theory* Ser. B **13** (1972), 95–98.

[4] D. J. Rose, R. E. Tarjan, G. S. Leuker - Algorithmic aspects of vertex elimination on graphs, *SIAM J. Comput.* 5 (1976), 266-283.

[5] F. Roussel, I. Rusu - A linear algorithm to color i-triangulated graphs, Research Report RR-98-11, LIFO, Université d'Orléans, France (1998).