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Cutsets in perfect and minimal imperfect graphs

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CUTSETS IN PERFECT AND MINIMAL IMPERFECT GRAPHS

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A set of vertices S is called a *cutset* of a graph $G = (V, E)$ if $V - S$ induces in G a disconnected graph. In the perfect graph theory, the idea of breaking a graph into smaller (and, preferably, easier to deal with) parts is natural, and probably the first promising attempt is to break the graph into connected components. A very simple reasoning then shows that a graph which admits such a decomposition cannot be minimal imperfect. By Lovász's perfect graph theorem [43], a graph whose complement admits such a decomposition cannot be minimal imperfect neither. Consequently, our simple initial remark yields, on the one hand, a certificate for a graph not to be minimal imperfect and, on the other hand, an hereditary class of perfect graphs (containing those graphs in which every induced subgraph either is non-connected or has non-connected complement graph).

This is only the first one of a series of results involving cutsets (in the previous example, we could imagine that we have an empty cutset in the graph). As suggested before, cutsets turned out to be interesting in perfect graph theory mainly with respect to two kind of applications.

First, sufficient conditions for a graph *not* to be minimal imperfect can be found, sometimes yielding easy (from a theoretical point of view) and/or efficient (from a practical point of view) tests. The classical follow-up to this type of results is to define new (or characterize old) classes of perfect graphs by imposing the sufficient condition on every induced subgraph of a graph in the class.

Secondly, composition/decomposition operations may be described. They are a good support for recursive reasonings and algorithms, as well as for defining new (or characterizing old) classes of perfect graphs by starting with basic graphs and successively composing. In the end, it would be desirable to find such basic graphs and such composition operations which would characterize the class of all perfect graphs.

Our aim here is to present the various aspects of cutsets in perfect and minimal imperfect graphs; this will be done according to the following plan:

1. How did it start?
2. Main results on minimal imperfect graphs
3. Applications: star cutsets
4. Applications: clique and multi-partite cutsets
5. Applications: stable cutsets
6. Two (already solved) conjectures
7. The connectivity of minimal imperfect graphs
8. Some (more) problems

For the definitions and notation not given here, the reader is referred to [4].

1 How did it start?

Let us consider two graphs $G_1 = (V_1, E_1), G_2 = (E_2, V_2)$, and assume that each of them contains a given graph F as an induced subgraph. Denote by $W_1 \subseteq V_1$, respectively $W_2 \subseteq V_2$ the set of vertices in G_1, respectively G_2 which induce F, and let $f : W_1 \to W_2$ be an isomorphism between the two induced subgraphs. The *F-bonding* (or *F-identification*) of G_1 and G_2 is the identification, for every $x \in W_1$, of x with $f(x)$ (yielding a new vertex adjacent to all the neighbours of x in G_1 and of $f(x)$ in G_2). A new graph results, denoted $G = G_1 \Phi_F G_2$, It is an easy remark that the vertex set W obtained from identification of W_1 and W_2 is a cutset in G.

It is an easy observation (see [23], [2]) that clique bonding (in the definition above, F is a clique) preserves perfection, that is, the graph G obtained by clique bonding of two perfect graphs G_1, G_2 is also perfect. As a by-product, no minimal imperfect graph can contain a clique cutset. Moreover, Tucker [62] proved that minimal imperfect graphs cannot have cutsets of type $\{x\} \cup N(x)$ neither, where x is a vertex and $N(x)$ its *neighbourhood*, that is, the set of all its neighbours in the graph. Observe that this result does not imply that the corresponding bonding operation preserves perfection.

In fact, in [15] a much more general result is shown: there is no graph F, except for the cliques, for which F-bonding preserves perfection. And it is easy to imagine why: given two non-adjacent vertices x, y of F, it is sufficient to exhibit two graphs G_1, G_2 containing F as an induced subgraph and such that x, y are joined by an odd (respectively even) chordless path in G_1 (respectively in G_2) whose only vertices in F are x, y. The F-bonding of G_1 and G_2 would then give a graph containing at least one odd chordless cycle, i.e. an imperfect graph. This can be done as follows:

- the complement graph $\overline{G_1}$ of G_1 is obtained by identifying the two vertices x, y in $\overline{F}$ with, respectively, the two vertices a, b of the graph in Fig. 1. As shown in [54], the new graph is perfect (and obviously a chordless path of length three joins x to y).
- G_2 is obtained from F by adding a new universal vertex.

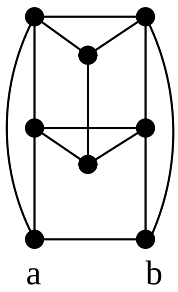

Figure 1:

Fortunately, while the strong affirmation "F-bonding preserves perfection" is false for every non-complete graph F, the weaker affirmation "no minimal imperfect graph contains a cutset which induces F" can still be true for some F (and, in fact, it is sufficient to consider minimal imperfect graphs different from the odd holes and odd anti-holes).

2 Main results on minimal imperfect graphs

From a structural point of view, three types of cutsets are known, which cannot appear in a minimal imperfect graph different from a hole (for the numerical point of view see Section 7). The first one (in the order of their publication) concerns *stable* cutsets, i.e. cutsets S such that the subgraph $G[S]$ induced by S in G is edgeless. It was proved in [61]:

Fact 1 (Tucker) *No minimal imperfect graph has a stable cutset, except for the odd holes.*

Furthermore, Chvátal [8] defined a *star cutset* to be a cutset S containing some vertex x which is adjacent to every other vertex in S (x is then called *universal* in $G[S]$). Then we have the following statement, known as the Star-Cutset Lemma:

Fact 2 (Chvátal) *No minimal imperfect graph has a star-cutset.*

The third result was proved by Cornuéjols and Reed in [16]. A *complete multi-partite graph* is a graph whose vertex set may be partitioned in $k \geq 1$ stable sets $U_1, U_2, \ldots, U_k$ such that all the edges exist between U_i, U_j for every $i \neq j$ in $\{1, 2, \ldots, k\}$. Then, we can define a *complete multi-partite cutset* as a cutset S for which $G[S]$ is a complete multi-partite graph.

Fact 3 (Cornuéjols & Reed) *No minimal imperfect graph has a complete multi-partite (not a stable) cutset.*

It is worth noticing that, although they are expressed similarly to each other here above, these three results have been proved in different (and stronger, for the first two of them) forms. The approach (by contradiction) is the same: given the cutset S of $G = (V, E)$ and a connected component V_1 of $G[V - S]$, then two graphs $G_1 = G[V_1 \cup S]$, $G_2 = G[V - V_1]$ are defined, and they are both supposed to be k-colourable ($k \geq 1$). Then, a stable set U is searched with the property that $G - U$ is $(k-1)$-colourable (if it is found, then G is k-colourable). In the case of a stable cutset, U is found without even imposing the condition that G is minimal imperfect; G has only to be odd hole-free. For a star-cutset, the result is weaker: k is taken to be equal to $\omega(G)$; but, once more, U is always found and the proof doesn't use other properties related to the minimal imperfection of G (then the ω-colourability of G_1 and G_2). Finally, for a multi-partite cutset, k is taken equal to $\omega(G)$, but U is not always found, even if the hypothesis that G is minimal imperfect is used. Then, in the first two cases the contradiction is easy to obtain, even for a partitionable graph. In the third one, additional properties of minimal imperfect graphs are needed to conclude.

Fact 2 and Fact 3 share another common property; they are particular cases of the Skew partition conjecture proposed in [8]. We say that a graph $G = (V, E)$ admits a *skew partition* if its vertex set V can be partitioned into nonempty sets A, B, V_1, V_2 such that:

- every vertex in A is completely adjacent to every vertex in B (we say that $S = A \cup B$ is the *complete join* of A, B);
- $V - S = V_1 \cup V_2$ such that no edge exists between V_1 and V_2.

Then the Skew partition conjecture claims that:

Conjecture 1 (Chvátal) *No minimal imperfect graph admits a skew partition.*

The interest of such an affirmation (if proved) comes from the self-complementarity of a skew partition: G admits a skew partition if and only if $\overline{G}$ admits a skew partition. If we recall (on the one hand) that, by Lovász's perfect graph theorem, G is perfect if and only if $\overline{G}$ is, and (on the other hand) that attempts are made to construct the class of perfect graphs from some basic graphs using some composition operations, we easily see that the skew partition fulfills more than one of the features a good operation should fulfill.

Although *this* generalization of the star-cutset lemma is not yet proved or disproved, such generalizations exist, due to Olariu [46] and Hoàng [35]. For a graph $G = (V, E)$ with cutset S, let V_1 be a connected component of $G[V - S]$ and let V_2 stand for the set of vertices $V - S - V_1$. Denote $G_1 = G[V_1 \cup S]$, $G_2 = G[V_2 \cup S]$ and, for a coloring f of G and every $U \subseteq V$, let $f(U)$ be the set of colours in U.

Fact 4 (Olariu) *No minimal imperfect graph contains a cutset $S = A \cup B$ with the properties*

(O1) $A \cap B = \emptyset, A \neq \emptyset$;
(O2) there exist optimal colourings $\mathcal{C}_1, \mathcal{C}_2$ of G_1, G_2 respectively, such that

$$\mathcal{C}_i(A) \cap \mathcal{C}_i(B) = \emptyset,\ i = 1, 2$$
$$|\mathcal{C}_1(A)| = |\mathcal{C}_2(A)| < \omega(G)$$

A cutset with properties (O1), (O2) is called a *partitionable* cutset. It is easy to see that the Star-cutset Lemma is a particular case of Fact 4 (just take $A = \{x\}$, where x is the universal vertex of S). Moreover, the most naïve attempt to prove perfection results using Fact 4 yields an interesting connection with another coloring problem.

To see this, consider a graph G whose all proper induced subgraphs are perfect, and for which we want to prove the perfection using Fact 4. We denote S a cutset of G, and

G_1, G_2 as before. As G_1 is a perfect graph, we can find an ω-colouring $\mathcal{C}_1$ of G_1. Then, we can try to apply Olariu's result by considering a subset $\mathcal{A}$ of colours in $\mathcal{C}_1$ such that $|\mathcal{A}| < \omega$), and the set $A \subseteq S$ of those vertices in S whose colour is in $\mathcal{A}$. Now, we have to colour G_2 with ω colours such that the vertices in A (respectively $B = S - A$) have colours in $\mathcal{A}$ (respectively not in $\mathcal{A}$). Notice that it is not *requested* that every vertex has *the same* colour in G_1 and G_2, but only that the colour belongs or not to $\mathcal{A}$.

On the contrary, we can even request that *at least* one vertex in S changes its colour. Otherwise, the resulting colouring (if found) is an ω-colouring of G and Olariu [46] shows that in this case S can be seen as a partitioned cutset $S = A' \cup B'$ such that A' contains the vertices of *one* fixed colour; this case can then be included in the case $|\mathcal{A}| = 1$ above. And, in fact, in the case $|\mathcal{A}| = 1$ we know to solve the problem, i.e. to answer the question whether G_2 can be coloured with ω colours such that the vertices in A have colours in $\mathcal{A}$ and the vertices in B have colours not in $\mathcal{A}$. This can be done using the theorem and the polynomial algorithm in [42].

In the general case where $|\mathcal{A}|$ is arbitrary, the problem is a particular case of the following one:

The list colouring problem. Let G be a graph and assume that each vertex v in G has assigned a list $L(v)$ of possible colours. Is there a colouring of G such that every vertex v has a colour in $L(v)$?

Proposed independently in [19] and [64], this problem is NP-complete (see [63] for a survey).

We give now Hoàng's partial result [35] on the Skew partition conjecture, which is another generalization of the Star-cutset lemma (case $A = \{x\}$, where x is the universal vertex of S):

Fact 5 (Hoàng) *No minimal imperfect graph admits a skew partition A, B, V_1, V_2 with the property*

(H) there exist optimal colourings $\mathcal{C}_1, \mathcal{C}_2$ of G_1, G_2 respectively, such that

$$|\mathcal{C}_1(A)| \geq |\mathcal{C}_2(A)|$$
$$|\mathcal{C}_1(B)| \geq |\mathcal{C}_2(B)|$$

(as before, if $S = A \cup B$, then $G_1 = G[V_1 \cup S]$, while $G_2 = G[V_2 \cup S]$)

This statement is used to prove two other particular cases of the Skew partition conjecture. In the graph G with the skew partition A, B, V_1, V_2, the set $S = A \cup B$ is a *U-cutset* if there are distinct vertices $u_1, u_2 \in V_1$ such that $N(u_1) \supseteq A$ and $N(u_2) \supseteq B$. The set S is a *T-cutset* if there exist distinct vertices $u_1 \in V_1, u_2 \in V_2$ such that $N(u_1) \supseteq A$ and $N(u_2) \supseteq A$.

Fact 6 (Hoàng) *No minimal imperfect graph has a skew partition A, B, V_1, V_2 such that $S = A \cup B$ is a U-cutset (respectively, a T-cutset).*

As an application, Hoàng shows that a graph G whose odd cycles of length at least five have two or more chords (also called a *Meyniel graph*) either is bipartite, or $\overline{G}$ has a star- or U-cutset.

3 Applications: star-cutsets

Since they were defined by Chvátal, the star-cutsets are probably the tool the most frequently used to prove perfection. In its paper, Chvátal [8] already notices some of these applications: clique bonding preserves perfection; substitution (i.e. replacing a vertex v of G with a graph H whose vertices have in G the same neighbours as v) preserves perfection; amalgam (see [6]) preserves perfection. Even the (very) particular case of clique cutsets has so many applications that we have to treat it in a particular section (Section 4).

Given a class $\mathcal{G}$ of graphs and P a predicate, define the *closure of $\mathcal{G}$ under P* (denoted $\mathcal{G}^P$) recursively by the rules:

(i) if $G \in \mathcal{G}$ then $G \in \mathcal{G}^P$;

(ii) if G satisfies P, and if $G - v \in \mathcal{G}^P$ for every vertex v in G, then $G \in \mathcal{G}^P$.

It is easy to see (by induction) that, whenever P is a property a minimal imperfect graph *cannot* have, the perfection of every graph in $\mathcal{G}$ implies the perfection of every graph in $\mathcal{G}^P$. Chvátal considered this definition in the particular case of the predicate (denoted $*$): "G or $\overline{G}$ has a star-cutset". He denoted by TRIV the class of graphs with at most two vertices and by BIP the class of all bipartite graphs, and noticed that both TRIV* and BIP* are involved in nice properties of perfect graphs. For instance, Hayward [30] proved that TRIV* is exactly the class of weakly triangulated graphs, i.e. graphs containing no cycle of length at least five (denoted C_k, $k \geq 5$) and no complement of such cycle. Some years later [28], he improved the "$\supseteq$" part of this statement by showing this important (when thinking to the strong perfect graph conjecture) property of unbreakable graphs (a graph G is *breakable* if either G or $\overline{G}$ has a star cutset, and *unbreakable* in the contrary case).

Fact 7 (Hayward) *In an unbreakable graph, every vertex is contained in a C_k or $\overline{C}_k$, $k \geq 5$.*

The identity between TRIV* and the class of weakly triangulated graphs is the only equality result proved so far. Many other theorems have been found which give inclusion relations, as indicated in the table below.

Class	Rel.	Class*	Proved by	Ref.
weakly triangulated	$=$	TRIV*	Hayward	[30]
Meyniel	$\subset$	BIP*	Chvátal	[8]
perfectly orderable	$\subset$	BIP*	Chvátal	[8]
opposition	$\subset$	BIP*	Olariu	[47]
alternately orientable	$\subset$	BIP*	Hoàng	[33]
2-coloured, odd P_4	$\subset$	(BIP $\cup\overline{\text{BIP}}$)*	Hoàng	[34]
2-coloured, even P_4	$\subset$	(Berge $K_{1,3}$-free)*	Chvátal, Hoàng	[10]
2-coloured, partners	$\subset$	(BIP $\cup\overline{\text{BIP}}$)*$\cup$(Berge $K_{1,3}$-free)*	Chvátal	[11]
Berge, $P(G)$ K_3-free	$\subset$	(BIP $\cup\overline{\text{BIP}}$)*	Hayward, Lenhart	[31]
$(bull, P_5, \overline{P_5})$-free, ≥ 6	$\subset$	(BIP $\cup\overline{\text{BIP}}$)*	Fouquet	[21]
pan-free	$\subset$	($K_{1,3}$-free)*	Olariu	[48]
$(P_5, K_{2,3})$-free	$\subset$	($K_{1,3}$-free)*	de Simone, Galluccio	[17]
slim	$\subset$	(strict quasi-parity)*	Hertz	[32]

The definitions not given here and the proofs may be found respectively in the papers indicated in the last column. The class $\overline{\text{BIP}}$ contains all the complements of bipartite graphs.

We leave for a moment the star-cutset domain in order to make a remark on partitionable cutsets. With the predicate π: "G is either a clique, or a stable set, or else G has a partitionable cutset" we obtain (see [46]) the following characterization of perfect graphs:

Fact 8 (Olariu) *The class of perfect graphs is exactly the class* TRIV$^\pi$.

The proof of this claim is easy and relies on the fact that if G is a perfect graph (not a clique, not a stable, of at least three vertices) with cutset S and a given coloring, then the vertices of S of a fixed colour can be chosen to form the set A. Thus, *every* cutset of G has the properties (O1) and (O2), while in a minimal imperfect graph *no* cutset has this property. Perfect graphs and minimal imperfect graphs can, therefore, be seen as extreme classes with respect to Olariu's property.

Let us come back to star-cutsets and in particular to their algorithmic aspects. The same paper of Chvátal [8] provides us with an algorithm to test whether a graph G has or not a star-cutset. It can be easily deduced from the following statement (a vertex v is said to *dominate* a vertex w if $\{v\} \cup N(v)$ contains $N(w)$):

Fact 9 (Chvátal) *A graph $G = (V, E)$ has a star-cutset if and only if it has at least one of the properties:*

a) there exists $w \in V$ such that $\{w\} \cup N(w)$ is a cutset;

b) G is not a clique and there exist adjacent vertices $v, w \in V$ such that v dominates w.

Then a graph G is breakable if and only if either G or $\overline{G}$ has the property a) or b). In fact, Chvátal (see Hayward [29]) proved a stronger property for graphs with more than four vertices:

Fact 10 (Chvátal) *Let G be a graph with at least five vertices. Then G is breakable if and only if G or $\overline{G}$ has property a).*

Now, though we can test in polynomial time whether G has a star-cutset, we cannot deduce a polynomial algorithm to test the belonging of a graph G to a class $\mathcal{G}^*$ (where $\mathcal{G}$ is assumed to be recognizable in polynomial time). If $\mathcal{G}$ is hereditary, then the following algorithm presented in [8] realizes (not necessarily in polynomial time) the indicated test. With the notation $F = H$ (if H has a star-cutset in step 4) or $F = \overline{H}$ (if $\overline{H}$ has a star-cutset in step 4), the sets V_1, V_2 in step 4 are as usual: V_1 is a connected component of $F - S$ while V_2 is $F - S - V_1$.

1. $L := \{G\}$.
2. If $L = \emptyset$, then return "$G \in \mathcal{G}^*$"; else remove some H from L.
3. If $H \in \mathcal{G}$ then goto 2.
4. If H or $\overline{H}$ has a star-cutset $S = \{v\} \cup V_0$, then
   $L := L \cup \{H - v, H - V_1, H - V_2\}$; goto 2
5. return "$G \notin \mathcal{G}^*$".

For a graph containing a lot of chordless cycles (take for instance the graph obtained from a chordless cycle on p vertices by substituting every vertex with a clique on two vertices), the number of operations needed by the execution of the algorithm can be very large if every chordless cycle is examined in step 3.

4 Applications: clique and multi-partite cutsets

This chapter is dedicated to the study of three classes of graphs, all of them contained (strictly or not) in some class $\mathcal{G}^c$, where c is the predicate "G has a clique cutset" and $\mathcal{G}^c$ is the closure of $\mathcal{G}$ under c. They are called triangulated graphs, i-triangulated graphs, clique separable graphs, and are included in each other in this order. They all have special vertices yielding elimination schemes, and for at least two of them the special vertices can be found using *lexicographic breadth first search* (abbreviated LexBFS), an algorithm proposed in [51] as a stronger version of the classical breadth-first search algorithm.

To describe LexBFS, assume that each vertex has a (initially empty) label which consists in a set of integers listed in decreasing order. The labels may be compared using dictionary (or lexicographic) order.

Algorithm LexBFS

Input: An arbitrary graph $G = (V, E)$.
Output: A one-to-one function $\sigma : \{1, 2, \ldots, n\} \to V$ (an order $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ on V).

begin
assign the label $\emptyset$ to each vertex;
for $i := n$ downto 1 do
pick an unnumbered vertex v with largest label (in lexicographic order);
$\sigma(i) := v$; {comment: this assigns to v the number i}
for each unnumbered vertex $w \in N(v)$ do
add i to the label of w (at the end)
end

Consider first the class of triangulated graphs. A graph is *triangulated* if it contains no chordless cycle C_k, $k \geq 4$. In a triangulated graph every minimal cutset is a clique. Moreover, using results in [18], [22] we have the following (a vertex is said *simplicial* in a subgraph H if its neighbours in H induce a clique):

Fact 11 (Dirac; Fulkerson & Gross) *The following three statements are equivalent for a graph $G = (V, E)$ with n vertices:*
1. *G is triangulated;*
2. *every minimal cutset induces a clique;*
3. *G has a perfect elimination scheme, i.e. an order $[v_1, v_2, \ldots, v_n]$ of its vertices such that v_i is simplicial in $G[v_i, v_{i+1}, \ldots, v_n]$. Moreover, if G is not a clique, then it contains two non-adjacent simplicial vertices.*

Using the equivalence between statements 1, 2 and the definition of $\mathcal{G}^c$ (recall that c is the predicate "G has a clique cutset"), it is easy to deduce that if $\mathcal{K}$ is the set of all cliques, then the class of triangulated graphs is exactly $\mathcal{K}^c$. The algorithm LexBFS is a good (since efficient) tool to find the perfect elimination scheme in statement 3: starting with an arbitrary unnumbered vertex v, a perfect elimination order $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ is found (see [51]). Then $v_1 = \sigma(1)$ is a simplicial vertex in G and a new execution of LexBFS, this time starting with v_1, will obtain another perfect elimination order $[\sigma'(1), \sigma'(2), \ldots, \sigma'(n) = v_1]$ whose vertex $w_1 = \sigma'(1)$ is again a simplicial vertex (non-adjacent to v_1 as long as v_1 is non-universal). LexBFS is used on the one hand for giving a linear recognition algorithm for triangulated graphs (see [51]), and on the other hand to obtain a coloring algorithm for these graphs (by performing a greedy coloring with the order $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$).

This last property is still true for a larger class of graphs, the i-triangulated graphs. A graph is called *i-triangulated* if every odd cycle with five vertices or more has at least two non-crossing chords. Their perfection was proved very early in [23] using clique cutsets, but multi-partite cutsets also play an important role in their structure, as shown in [52].

To see this, say that a graph is *of type 1* if it is the complete join between a connected bipartite graph and a clique, and *of type 2* if it is a complete multi-partite graph. Also, call a vertex *m-simplicial* in a subgraph H if its neighbours in H form a complete multi-partite graph.

Fact 12 (Gallai; Roussel & Rusu) *The following statements hold for every i-triangulated graph $G = (V, E)$ with n vertices:*

1. *either G has a clique cutset, or is of type 1, or else is of type 2;*
2. *G has an m-perfect elimination scheme, i.e. an order $[v_1, v_2, \ldots, v_n]$ of its vertices such that v_i is m-simplicial in $G[v_i, v_{i+1}, \ldots, v_n]$. Moreover, if G is not a clique, then it contains two non-adjacent m-simplicial vertices.*

Statement 1 immediately implies that i-triangulated graphs are in the class $\mathcal{T}^c$, where $\mathcal{T}$ is the set of graphs of type 1 or 2. However, this is not a characterization of i-triangulated graphs. The existence of an m-perfect elimination scheme is not a characterization neither, thus it doesn't allow us to find a linear recognition algorithm, as it was the case for the preceding class (an $O(mn)$ recognition algorithm is proposed in [53]). However, LexBFS is still the efficient tool to find in linear time an m-perfect elimination scheme, and the order $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$ can be used to perform a greedy coloring (see [52]). It is worth noticing that in every such colouring, at the precise moment when a vertex $\sigma(i)$ is coloured, its already coloured neighbourhood (which is a complete multi-partite graph) has all the vertices in a stable set of the same colour. This strengthens the similarity with triangulated graphs, where every such stable set has a unique vertex (obviously coloured with only one colour).

The graphs in $\mathcal{T}^c$ are called *clique separable graphs*. Gavril [25] defined them and gave the first polynomial recognition algorithm (the best one known nowadays was given by Tarjan [60]). This graphs have an elimination scheme too, defined in [44] using the notion of *pretty* vertex, which is a vertex whose neighbourhood in the graph induces a $(P_4, 2K_2)$-free graph. Then it can be shown that:

Fact 13 (Maffray & Porto & Preissmann) *Every clique separable graph G with n vertices has a pretty elimination scheme, i.e. an order $[v_1, v_2, \ldots, v_n]$ of its vertices such that v_i is pretty in $G[v_i, v_{i+1}, \ldots, v_n]$. Moreover, if G is not a clique, then it contains two non-adjacent pretty vertices.*

Triangulated and clique separable graphs can be recognized in polynomial time using Whitesides' $O(mn)$ algorithm [66] for finding clique cutsets in arbitrary graphs (but less efficiently that the best known algorithm). As Gavril [25] showed that the tree built by successively breaking the graph G using clique cutsets (called a *clique cutset tree*) has $O(n^2)$ vertices, the indecomposable subgraphs of G can be found in $O(mn^3)$. Then it remains to test whether the indecomposable subgraphs are in $\mathcal{K}$ (for triangulated graphs), respectively in $\mathcal{T}$ (for clique separable graphs). Thus the efficiency of recognition algorithm depends on the efficiency of building the clique cutset tree, and on the efficiency to test whether a graph belongs or not to one of the two classes. As shown by Tarjan [60], the former can be realized in $O(mn)$ (while the latter needs less than that), thus recognizing clique separable graphs takes only $O(mn)$ time.

An important difference appears between multi-partite cutsets and star- or clique cutsets: testing whether a graph has a multi-partite cutset is NP-complete [41]. And the same holds for stable cutsets (see [41]), that we approach in the next section.

5 Applications: stable cutsets

Stable cutsets are much less encountered in perfection proving, or in algorithm designing than clique cutsets. The predicate s: "G has a stable cutset" seems to be much more uncomfortable to use than "G has a clique cutset", and the only such applications that came into our attention are either extremely easy or too intimately related to some specific properties of the graph.

In the first category enters the observation that the class of triangle-free graphs is exactly TRIV^s: indeed, every triangle-free graph with at least three vertices has a non-universal vertex, and its neighbourhood is a stable cutset of the graph (by induction we obtain the desired conclusion); conversely, if we assume by contradiction that a graph containing a triangle belongs to TRIV^s, then because of condition ii) in the definition of TRIV^s we deduce that the triangle has a stable cutset, a contradiction. Moreover, triangle-free graphs are exactly the class of graphs possessing a stable elimination order, i.e. an order $[v_1, v_2, \ldots, v_n]$ of vertices such that the neighbourhood of v_i in $G[v_i, v_{i+1}, \ldots, v_n]$ is a stable set.

To illustrate the second category, we present some results that have been obtained by Corneil, Fonlupt [14] on stable bonding. As we indicated in Section 1, only clique bonding preserves perfection, for the other graphs F examples exist of perfect graphs whose F-bonding does not yield a perfect graph. In the case where the vertices of F form a stable set, Fact 1 insures that only odd holes can be generated, not odd anti-holes. The example we gave in Section 1 to prove that F-bonding does not preserve perfection suggests that perfection could however be preserved for graphs with particular properties related to path parity.

Therefore, in [14], a *direct chain* (for some stable cutset S) in $G = (V, E)$ is defined to be a chordless path whose internal vertices are in $V - S$. Then a graph G is said to satisfy the *strong chain condition* (SCC) on the stable cutset S if:

1. G is connected and for every pair of vertices $v, w \in S$ there exists at least one chordless direct chain.

2. for every pair of vertices $v, w \in S$ all chordless chains with endpoints v, w have the same parity (denoted $sign(v, w; G)$, equal to 1 if the chain is odd and to 0 otherwise).

Then consider two graphs G_1, G_2 with stable sets S_1, respectively S_2 such that $|S_1| = |S_2|$. Perform a one-to-one identification of the vertices in S_1 with the vertices in S_2 and call S the unique stable set (of size identical to $|S_1|$ and $|S_2|$) obtained in this way. Then we have:

Fact 14 (Corneil & Fonlupt) *If G_i ($i = 1, 2$) are perfect graphs that satisfy SCC on S and for every pair of vertices $v, w \in S$ we have $sign(v, w; G_1) = sign(v, w; G_2)$, then the graph G obtained by S-bonding of G_1 and G_2 is also perfect.*

It could seem interesting to release (if possible) condition 2 by considering only chordless *direct* chains instead of chordless chains. The example in Fig. 2 (given in [14]) shows that this cannot be done (the set $S = \{s_1, s_2, s_3\}$ satisfies the hypothesis of Fact 14 with condition 2 released, but the resulting graph is not perfect since $as_1bcdes_3fs_2a$ is an odd hole).

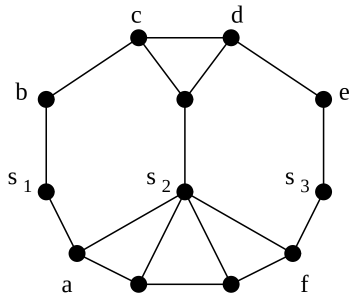

Figure 2:

Notice that testing condition 2 for an *arbitrary* graph G is a co-NP-complete problem. The complementary problem, i.e. given G, S test whether S contains two vertices joined by chordless chains of different parity, has as a particular case (when $|S| = 2$) the so-called *Path parity problem*:

Path parity problem. Given $G = (V, E)$ and two vertices x, y, determine if there exist chordless paths (or chains) of different parity connecting them.

This problem is NP-complete [3] for *arbitrary* graphs, but for *perfect graphs* (this is the case we are interested in) its complexity is not known (partial answers may be found in [1], [40], [56]). However, it is known that this problem is polynomial *if* testing the perfectness of a graph G is polynomial (see [14]).

As long as this question is not answered, it is tempting to find sufficient conditions for condition 2. to be true, as follows. Let G_1, G_2 have stable cutsets S_1, S_2 respectively, and assume that each $G_i - S_i$ contains a connected component K_i which is a clique, such that $|K_1| = |K_2|$. Moreover, suppose that every vertex $s \in S_i$ has exactly one neighbour in K_i ($i = 1, 2$) and that an isomorphism f exists between $G_1[K_1 \cup S_1]$ and $G_2[K_2 \cup S_2]$. Then (see [14]) the graph resulting by stable bonding of $G_1 - K_1$ and $G_2 - K_2$ using S_1, S_2 and the isomorphism f is perfect. The only condition whose testing could be delicate here is the isomorphism condition on $G_1[K_1 \cup S]$ and $G_2[K_2 \cup S]$, but the graphs are simple enough to allow us a polynomial verification (in fact, it is sufficient to compare the lists of degrees of the two graphs).

6 Two (already solved) conjectures

In the preceding sections, we evoked some attempts to prove the strong perfect graph using composition/decomposition operations. This is one of the reasons which make the skew partition conjecture interesting, which motivate the definition of the classes $\mathcal{G}^P$, which push the researches through the discovering of new properties that minimal imperfect graph cannot have. All these properties, old or new, could possibly be put together in a strong unique predicate P such that $\mathcal{G}^P$ would contain all Berge graphs, for some set of perfect basic graphs $\mathcal{G}$ suitably chosen. (A graph is called *Berge* if it has no odd hole and no odd anti-hole).

Unfortunately, the properties of minimal imperfect graphs we know nowadays are not sufficient to deduce such a result, or else the class of basic perfect graphs we need has not yet been identified. The two conjectures below have both been invalidated, and the counter-examples are small enough to make us believe that the properties that are brought together are not sufficiently strong. We preferred to formulate them without clearly specifying $\mathcal{G}$ and P, but the reader will have no difficulty to identify the predicate and the basic graphs. The first one was proposed by Reed [50] (the *linegraph* of a graph $H = (X, U)$ has a vertex for every edge in U and two vertices are adjacent if and only if the corresponding edges share a vertex).

Conjecture 2 *Let G be a Berge graph such that*

1. *neither G nor $\overline{G}$ has a star-cutset;*
2. *neither G nor $\overline{G}$ has an even pair.*

Then G or $\overline{G}$ is the linegraph of a bipartite graph.

Hougardy [37] gave a counter-example on 20 vertices, and noticed that even if G is C_4-free the conjecture remains false.

The second conjecture (proposed by Hoàng, see [37]) has stronger hypothesis and weaker conclusions, but is still false, as proved in [55].

Conjecture 3 *Let G be a Berge graph such that*

1. *neither G nor $\overline{G}$ has a star-cutset;*
2. *neither G nor $\overline{G}$ has a stable cutset;*
3. *neither G nor $\overline{G}$ has an even pair.*

Then G or $\overline{G}$ is diamond-free.

It can be shown (see [55]) that the counter-example to Conjecture 3 may be grown to give counter-examples to each weaker conjecture obtained by replacing the diamond with the complete join of a clique and a stable set. Moreover, it remains a counter-example even if we add the following hypothesis (deduced from the Odd pair conjecture, which is neither proved nor invalidated):

4. *neither G nor $\overline{G}$ has an odd pair*

And, in fact, it has another discouraging property: even if the hypothesis

5. neither G nor $\overline{G}$ has a skew partition

is added to the conjecture, this one remains false and the same graph is a counter-example. Modifying the conjecture such that the same graph is not a counter-example no more seems therefore to ask for other properties of minimal imperfect graphs. One of them could be the one conjectured in [9], which involves disconnected cutsets (such cutsets can be found in the counter-example above):

Conjecture 4 *No minimal imperfect graph has a disconnected cutset, except for the odd holes.*

The versions of this conjecture where the cutset is P_4-free, or even P_3-free are open too. The latter case is equivalent to saying that the cutset is a union of vertex-disjoint cliques, and may be simplified by taking the case of only two cliques, for which no proof exists yet neither. Except for this last case, in all these problems the cutset may be asked to be the neighbourhood of some vertex, still yielding an unsolved statement.

7 The connectivity of minimal imperfect graphs

All along the paper we were interested in the graphs F such that no minimal imperfect graph (except for odd holes and odd anti-holes) has a cutset which induces F, and so far we approached this question with respect to the *structure* of F, paying no attention to the *cardinality* of F. This different aspect of the question was first treated by Olaru [49]:

Fact 15 (Olaru) *The minimum degree of a minimal imperfect graph is at least* $2\omega - 2$.

This result remained the only numerical estimation of (certain) cutsets in minimal imperfect graphs, until Hougardy [38] proved the following statement which involves all the cutsets of the graph:

Fact 16 (Hougardy) *The connectivity number of a minimal imperfect graph is at least* ω.

The best lower bound known today is due to Sebö [57], who had the intuition that the gap between the high clique rank of the minimal imperfect graph G and the low clique ranks of its perfect subgraphs G_1, G_2 (defined as in Section 2, with respect to some cutset S) should be the consequence of a big intersection of G_1 and G_2. Then a simple calculation yields the first part of the result below. This result is valid not only for a minimal imperfect graph but, more generally, for a *partitionable graph*, i.e. a graph $G = (V, E)$ such that:

(i) integers $\alpha \geq 2, \omega \geq 2$ exist with the property $|V| = \alpha * \omega + 1$;

(ii) for each $v \in V$, $G - v$ can be partitioned both into ω-cliques and into α-stable sets. In this case, ω, α are the classical parameters clique number and stability number for the partitionable graph G.

Fact 17 (Sebö) *If G is partitionable, then the connectivity number of G is at least $2\omega - 2$. Furthermore, if $S \subset V$ is a cutset of cardinality $2\omega - 2$, then:*

1. *$\omega(S) < \omega - 1$;*
2. *$G - S$ has exactly two connected components V_1, V_2;*
3. *$V_1 \cup S, V_2 \cup S$ induce uniquely colourable graphs.*

This theorem is tight for arbitrary $\omega \geq 2$ and $\alpha \geq 2$. To see this, call an (α, ω)-*web* a graph on $\alpha\omega + 1$ vertices so that $\omega(G) = \omega, \alpha(G) = \alpha$ and the vertices of V may be arranged in a cyclical order such that every set of ω consecutive vertices is an ω-clique. Now, consider a *normalized* (α, ω)-web, that is, an (α, ω)-web in which every edge is contained in some ω-clique. It is easy to see that two sets of $\omega - 1$ consecutive vertices each form a cutset iff the two sets are disjoint and not next to each other in the cyclic order. Every such cutset has $2\omega - 2$ vertices, and the neighbourhood of every vertex has this form.

Obviously, both in odd holes and in odd anti-holes the vertices have the degree $2\omega - 2$; consequently, as far as we know every minimal imperfect graph has this property. Thus, if the strong perfect graph conjecture is true, then the following statement also holds:

Conjecture 5 (Sebö) *In a minimal imperfect graph, the neighbourhood of every vertex is a cutset of cardinality $2\omega - 2$.*

Conversely, to deduce the strong perfect graph conjecture from this conjecture, it can be noticed that properties specific to minimal imperfect graphs and not always true for partitionable graphs should be used, since in a normalized (α, ω)-web (which is partitionable, but not minimal imperfect for $\omega > 2$ and $\alpha > 2$) the neighbourhood of every vertex has, as noticed before, cardinality $2\omega - 2$ and still this graph is not an odd hole or odd anti-hole. Sebö [57] proposes to make use of the property (see [13]) that minimal imperfect graphs have no *small transversal*, i.e. no set of at most $\alpha + \omega - 1$ which hits every ω-clique and every α-stable in the graph. To this, it can be added that, by Lovász's perfect graph theorem, a graph is minimal imperfect iff its complement is, therefore the preceding conjecture may be applied to both G and $\overline{G}$ (and this is sufficient to eliminate the (α, ω)-webs with $\alpha > 2, \omega > 2$).

Conjecture 6 (Sebö) *Let G be partitionable so that G (respectively $\overline{G}$) has a cutset of cardinality $2\omega - 2$ (respectively $2\alpha - 2$). Then G is either an odd hole, or an odd anti-hole, or else it contains a small transversal.*

Now, if Conjecture 5 and Conjecture 6 hold, then the strong perfect graph conjecture holds.

8 Some (more) problems

We close this survey of results on cutsets in perfect and minimal imperfect graphs by a list of open problems that do not include the conjectures discussed in the preceding sections. For some of them, references of partial results are given.

Problem 1 (Chvátal [9]) Design a polynomial algorithm to recognize graphs that admit a skew partition, or prove that the recognition problem is NP-complete.

Problem 2 (Corneil & Fonlupt [14]) Design a polynomial algorithm to test for the existence of a stable cutset in a perfect graph, or prove that the problem is NP-complete (and similarly for multi-partite cutsets).

Problem 3 (Chvátal [9]) Prove that graphs in BIP* are strict quasi-parity graphs, i.e. each subgraph which is not a clique contains an even pair. (partial results in [45])

Problem 4 (Hertz [32]) Call *slim graph* a graph obtained from a Meyniel graph by removing all the edges of a given induced subgraph. Is every slim graph in the class BIP*? (partial result in [36]).

Problem 5 (de Figueiredo & Hoàng [20]) Call *quasi-triangulated graph* a graph G such that, for every induced subgraph H of G, either H or $\overline{H}$ has a simplicial vertex. Characterize quasi-triangulated graphs by minimal forbidden induced subgraphs.

Problem 6 (Corneil & Fonlupt [14]) Design a polynomial algorithm to solve the path parity problem for perfect graphs, or prove that the problem is NP-complete.

Problem 7 (Ravindra [9]) Show that, in a minimal imperfect graph, the neighbourhood of a vertex of degree $2\omega - 2$ contains no stable set of cardinality 3.

Problem 8 (Sebö [57]) The *determined degree* of a vertex v is defined as the number of edges incident to v which are contained in ω-cliques. Let G be a partitionable graph such that G (respectively $\overline{G}$) has a vertex of determined degree $2\omega-2$ (respectively $2\alpha-2$). Prove that G is an odd hole, or an odd antihole, or else it has a small transversal.

References

[1] S. R. Arikati, C. Pandu Rangan, G. K. Manacher - Efficient reduction for path problems on circular-arc graphs, *BIT* 31 (1991) 182-193.

[2] C. Berge - Graphs and hypergraphs, North-Holland, Amsterdam 1973.

[3] D. Bienstock - On the complexity of testing for odd holes and odd induced paths, *Discrete Mathematics* 90 (1991), 85–92.

[4] A. Bondy, U. S. R. Murty - Graph Theory and applications, Elsevier North-Holland 1976.

[5] A. Brandstädt - Special graph classes, a survey, Universität Duisburg Gesamthochschule 1993.

[6] M. Burlet, J. Fonlupt - Polynomial algorithm to recognize a Meyniel graph, in *Topics on Perfect Graphs.* Math. Stud. 88 (C. Berge and V. Chvátal, eds.) (1984), 225–252.

[7] M. Burlet, J. Fonlupt - Polyhedral consequences of the amalgam operation, *Discrete Mathematics* 130 (1994), 39-55.

[8] V. Chvátal - Star-cutsets and perfect graphs, *Journal of Combinatorial Theory* B 39 (1985), 189–199.

[9] V. Chvátal - Problems concerning perfect graphs, 1st Workshop on perfect graphs, New Jersey, USA (1993).

[10] V. Chvátal, C. T. Hoàng - On the P_4-structure of perfect graphs I: even decompositions, *Journal of Combinatorial Theory* B 39 (1985), 209-219.

[11] V. Chvátal - On the P_4-structure of perfect graphs III: partner decompositions, *Journal of Combinatorial Theory* B 43 (1987), 349-353.

[12] V. Chvátal, N. Sbihi - Bull-free Berge graphs are perfect, *Graphs Combin.* 3 (1987), 127–139.

[13] V. Chvátal - An equivalent version of the strong perfect graph conjecture, *in Annals of Discrete Mathematics* 21 (1984) (C. Berge, V. Chvátal, eds.) (1984), 193-195.

[14] D. G. Corneil, J. Fonlupt - Stable set bonding in perfect graphs and parity graphs, *Journal of Combinatorial Theory* B 59 (1993) 1-14.

[15] D. G. Corneil, D. G. Kirkpatrick - Families of recursively defined perfect graphs, *in Proceedings of the 18th SE Conference* (1983) 237-246.

[16] G. Cornuéjols, B. Reed - Complete multi-partite cutsets in minimal imperfect graphs, *Journal of Combinatorial Theory* B 59 (1993) 191-198.

[17] C. de Simone, A. Galluccio - New classes of Berge perfect graphs, *Discrete Mathematics* 131 (1994) 67-79.

[18] G. A. Dirac - On rigid circuit graphs, Abh. Math. Sem Univ. Hamburg 25 (1961), 71–76.

[19] P. Erdös, A. L. Rubin, H. Taylor - Choosability in graphs, *Congressus Numerantium* XXVI (1979) 125-157.

[20] C. M. H. de Figueiredo, C. T. Hoàng - Problems on perfect graphs, Workshop on Perfect Graphs, Toronto (1998).

[21] J.-L. Fouquet - A decomposition for a class of $(P_5, \overline{P_5})$-free graphs, *Discrete Mathematics* 121 (1993) 75-83.

[22] D. R. Fulkerson, O. A. Gross - Incidence matrices and interval graphs, *Pacific Journal of Mathematics* 15 (1965) 835-855.

[23] T. Gallai - Graphen mit triangulierbaren ungeraden Vielecken, *Magyar Tud. Akad. Mat. Kutado Int. Közl.* 7, 3-36, 1962.

[24] M. R. Garey, D. S. Johnson - Computers and intractability, A guide to the theory of NP-completeness, W. H. Freeman and Company (1979).

[25] F. Gavril - Algorithms on clique separable graphs, *Discrete Mathematics* 19 (1977), 159-165.

[26] M. C. Golumbic - Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.

[27] M. Grötschel, L. Lovász, A. Schrijver - Polynomial algorithms for perfect graphs, *in Topics on Perfect Graphs.* Math. Stud. 88 (C. Berge and V. Chvátal, eds.) (1984), 325–356.

[28] R. B. Hayward - Discs in unbreakable Berge graphs, *Graphs and Combinatorics* 11 (1995), 249-254.

[29] R. B. Hayward - Murky graphs, *Journal of Combinatorial Theory* B 49 (1990), 200-235.

[30] R. B. Hayward - Weakly triangulated graphs, *Journal of Combinatorial Theory* B 39 (1985), 200–208.

[31] R. B. Hayward, W. Lenhart - On the P_4-structure of perfect graphs IV: Partner graphs, *Journal of Combinatorial Theory* 48 (1990) 135-139.

[32] A. Hertz - Slim graphs, *Graphs and combinatorics* 5 (1989), 149-157.

[33] C. T. Hoàng - Alternating orientation and alternating colouration of perfect graphs, *Journal of Combinatorial Theory* B 42 (1987), 264–273.

[34] C. T. Hoàng - On the P_4-structure of perfect graphs II: odd decompositions, *Journal of Combinatorial Theory* B 39 (1985), 220-232.

[35] C. T. Hoàng - Some properties of minimal imperfect graphs, *Discrete Mathematics* 160 (1996), 165-175.

[36] C. T. Hoàng, F. Maffray - On slim graphs, even pairs, and star-cutsets, *Discrete Mathematics* 105 (1992) 93-102.

[37] S. Hougardy - Counterexamples to three conjectures concerning perfect graphs, *Discrete Mathematics* 117 (1993), 245-251.

[38] S. Hougardy - Perfekte graphen diplomarbeit, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany (1991).

[39] W.-L. Hsu - Decomposition of perfect graphs, *Journal of Combinatorial Theory* B 43 (1987), 70–94.

[40] W. Hsu - Recognizing planar perfect graphs, *Journal of ACM* 34 (1987), 255-288.

[41] S. Klein, C. M. H. de Figueiredo - The NP-completeness of multi-partite cutset testing, *Congressus Numerantium* 119 (1996) 217-222.

[42] J. Kratochvíl, A. Sebö - Coloring precolored perfect graphs, *Journal of Graph Theory* 25 (1997), 207-215.

[43] L. Lovász - A characterization of perfect graphs, *Journal of Combinatorial Theory* Ser. B 13 (1972), 95–98.

[44] F. Maffray, O. Porto, M. Preissmann - A generalization of simplicial elimination orderings, *Journal of Graph Theory* vol. 23, no 2 (1996) 203-208.

[45] H. Meyniel - A new property of critical imperfect graphs and some consequences, *European Journal of Combinatorics* 8 (1987), 313–316.

[46] S. Olariu - A generalization of Chvátal's star-cutset lemma, *Information Processing Letters* 33 (1989/90), 301-303.

[47] S. Olariu - All variations on perfectly orderable graphs, *Journal of Combinatorial Theory* B 45 (1988), 150-159.

[48] S. Olariu - The strong perfect graph conjecture for pan-free graphs, *Journal of Combinatorial Theory* B 47 (1989), 187-191.

[49] E. Olaru - Zur Characterisierung perfekter graphen, *Elektronische Informationsverarbeitung und Kybernetik (EIK)* 9 (1973), 543-548.

[50] B. Reed - A semi-strong perfect graph theorem, Ph. D. Thesis, McGill University, Québec, Canada (1986).

[51] D. J. Rose, R. E. Tarjan, G. S. Leuker, Algorithmic aspects of vertex elimination on graphs, *SIAM J. Comput.* 5 (1976), 266-283.

[52] F. Roussel, I. Rusu - A linear algorithm to color i-triangulated graphs, Research Report n^o. RR-98-11, LIFO, Université d'Orléans (1998), submitted.

[53] F. Roussel, I. Rusu - Recognizing i-triangulated graphs in $O(mn)$, Research Report n^o. RR-98-10, LIFO, Université d'Orléans (1998), submitted.

[54] I. Rusu - Quasi-parity and perfect graphs, *Information Processing Letters* 54 (1995), 35-39.

[55] I. Rusu - Building counter-examples, *Discrete Mathematics* 171 (1997) 213-227.

[56] C. R. Satyan, C. Pandu Rangan - The path parity problem on some subclasses of perfect graphs, *Discrete Applied Mathematics* 68 (1996) 293-302.

[57] A. Sebö - The connectivity of minimal imperfect graphs, *Journal of Graph Theory* vol.23, no. 1 (1996) 77-85.

[58] A. Sebö - On critical edges in minimal imperfect graphs, *Journal of Combinatorial Theory* B 67 (1996), 62-85.

[59] P. Seymour - Problem Session, DIMACS meeting on polyhedral combinatorics, Morristown (1989).

[60] R. E. Tarjan - Decomposition by clique separators, *Discrete Mathematics* 55 (1985) 221-232.

[61] A. Tucker - Coloring graphs with stable cutsets, *Journal of Combinatorial Theory* B 34 (1983) 258–267.

[62] A. Tucker - Critical perfect graphs and perfect 3-chromatic graphs, *Journal of Combinatorial Theory* B 23 (1977) 143–149.

[63] A. Tuza - Graph colorings with local constraints - A survey, *Discussiones Mathematicae- Graph Theory* 17 (1997) 161-228.

[64] V. G. Vizing - Vertex colorings with given colors (in Russian), *Methody Discret Analiz*, 29 (1976) 3-10.

[65] S. H. Whitesides - A method for solving certain graph recognition and optimization problems with applications to perfect graphs, *in Topics on Perfect Graphs*, Mathematical Studies 88 (C. Berge and V. Chvátal, eds.), 1984, 281–298.

[66] S. H. Whitesides - An algorithm for finding clique cut-sets, *Information Processing Letters* 12 (1981), 31–32.