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Abstract

This paper generalizes previous works on perfectly orderable graphs by Olariu [11] and by Hoàng and al. [9]. Chvátal defined a graph to be perfectly orderable [2] if there exists a linear order $<$ on its set of vertices such that no induced path $abcd$ with edges ab, bc, cd has both $a < b$ and $d < c$. Given a graph G and a vertex v in G such that $G - v$ is perfectly orderable, we set some conditions on v for which we deduce that G is perfectly orderable. Our method allows to construct a new class of such graphs, recognizable in polynomial time, containing quasi-brittle graphs, charming graphs and some other classes of perfectly orderable graphs.

Keywords: graph, perfectly orderable graph, polynomial time.

Résumé

Dans ce rapport, nous généralisons des travaux antérieurs d'Olariu [11] et de Hoàng, Maffray, Olariu, Preissmann [9], sur les graphes parfaitement ordonnables. Chvátal a défini comme parfaitement ordonnable [2] tout graphe dont il existe un ordre linéaire $<$ sur l'ensemble de ses sommets tel qu'il n'existe pas de chemin induit $abcd$ avec les arêtes ab, bc, cd et ayant $a < b$, $d < c$. Étant donnés un graphe G et un sommet v de G tels que $G - v$ soit parfaitement ordonnable, nous posons des conditions sur v pour pouvoir déduire que G est parfaitement ordonnable. Notre méthode nous permet d'obtenir une nouvelle classe de graphes parfaitement ordonnables, reconnaissable en temps polynomial, contenant les graphes quasi-fragiles, les graphes charmants et d'autres classes de graphes parfaitement ordonnables.

Mots-clé : graphe, graphe parfaitement ordonnable, temps polynomial.

1 Introduction

A natural way to colour the vertices of a graph $G = (V, E)$ ordered in a sequence $v_1 < v_2 < \cdots < v_n$ by the set of "colours" $\{1, 2, 3, \cdots\}$ consists of scanning the sequence from v_1 to v_n and assigning to each v_j the smallest integer assigned to none of its neighbours v_i such that $i < j$. This way to colour the vertices of G is called *the greedy algorithm.* Certainly, this way may use a number of colours greater than the chromatic number of G. A graph G is said to be *perfectly orderable* [2] if there exists a linear order $<$ such that no induced P_4 $abcd$ in G has both $a < b$ and $d < c$ (such a P_4 is called an *obstruction* in $(G; <)$). Obviously, such an order, called *perfect order*, is also a perfect order on every $V' \subseteq V$, therefore the family of perfectly orderable graphs is hereditary. Recall that C. Berge defined a graph G to be *perfect* if the vertices of every induced subgraph H of G can be coloured with $\omega(H)$ colours, where $\omega(H)$ is the maximum clique size in H. It can be easily seen that a graph G is perfect if and only if every induced subgraph H contains a stable set intersecting every maximum clique in H. Berge and Duchet [1] defined a graph G to be *strongly perfect* if every induced subgraph H of G contains a stable set that intersects every maximal clique in H (as usual, "maximal" is meant with respect to set-inclusion). Chvátal [2] shown that perfectly orderable graphs are strongly perfect and that the greedy algorithm applied to a perfect order gives an optimal colouring.

A vertex v in a graph G is said to be *semi-simplicial* [7] if v is midpoint of no P_4 in G. A vertex v in a graph G is said to be *charming* [9] if v is not end-vertex in a P_5 in G, is not end-vertex in a P_5 in $\overline{G}$ and does not lie on a C_5 in G.

A *rooted graph* is a pair $H = (F, u)$ where F denotes a graph and u a vertex in F. Let $\mathcal{F}$ be a family of rooted graphs. A vertex v in a graph G is said to be $\mathcal{F}$-free if there is no induced subgraph F of G such that $H = (F, v)$ is isomorphic to a rooted graph of $\mathcal{F}$. For example, a semi-simplicial vertex is $\mathcal{F}$-free for the family $\mathcal{F} = \{(P_4, u)\}$ where u denotes a midpoint of P_4 , a charming vertex is $\mathcal{F}$-free for the family $\mathcal{F}$ described in Figure 1.

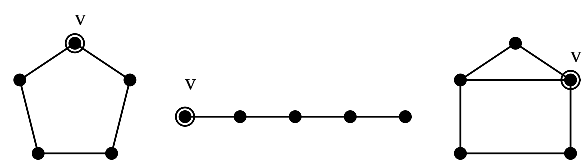

Fig. 1

Hoàng and al. [9] call *charming* any graph in which every induced subgraph has a charming vertex, and prove that every charming graph is perfectly orderable. A natural idea is to consider the following related notion: a vertex v in a graph G is said to be *nice* if v is not internal vertex in a P_5 in G, is not internal vertex in a P_5 in $\overline{G}$ and does not lie on a C_5 in G. A nice vertex is $\mathcal{F}$-free for the family $\mathcal{F}$ described in Figure 2. We call *nice* any graph in which every induced subgraph has a nice vertex, and we will prove that every nice graph is perfectly orderable (see below). In fact, this result is a corollary of Theorem 3 in Section 5.

An edge ab in a graph G is called a *symmetric wing* if there exist vertices c, d, p, q such

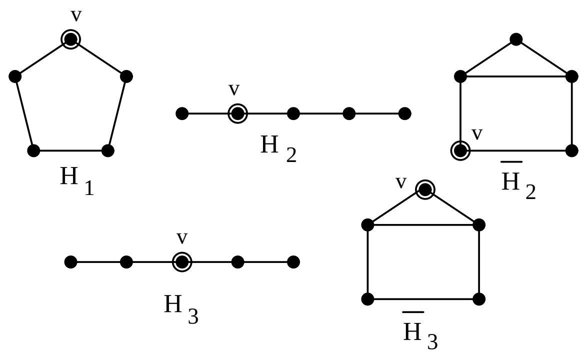

Fig. 2

that both *abcd* and *baqp* are induced P_4s in G. A vertex v in G is said to be *special* [11] if v is incident with no symmetric wing in G and $\overline{G}$.

Lemma 1 *An edge ab in a graph G is a symmetric wing if and only if ab is the edge whose ends are circled in the induced subgraphs of G depicted in Figure 3.*

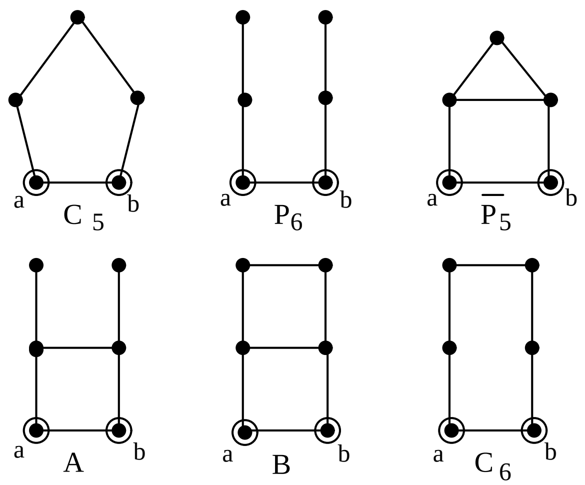

Fig. 3

Proof. Let c, d, p and q such that both *abcd* and *baqp* are P_4s in G. There are two cases according as $d = p$ or $d \neq p$. If $d = p$ then cq is the only one possible edge. If $d \neq p$ then cq, cp, dq, dp are possible edges. By examining every case, it is a routine matter to obtain the announced result. □

Lemma 2 *A vertex v in a graph G is special if and only if v is $\mathcal{F}_1$-free, where $\mathcal{F}_1$ is the family of rooted graphs $\{H_1, H_2, \overline{H_2}, H_4, \overline{H_4}\}$ (see Figures 2 and 4).*

Proof. By considering an end-vertex a of a symmetric wing and the rooted graphs (C_5, a), $(\overline{P_5}, a)$ and (A, a) in Figure 3 and their (rooted) complements, we obtain rooted graphs

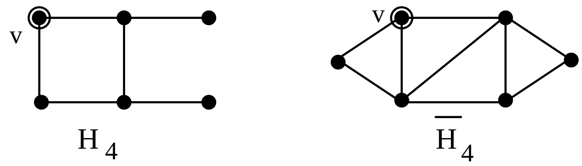

Fig. 4

isomorphic to H_1, H_2, $\overline{H_2}$, H_4 or to $\overline{H_4}$ (see Figures 2 and 4). The other graphs in Figure 3, that is (P_6, a), (C_6, a), (B, a), and their complements give rooted graphs containing a rooted graph (F, a) isomorphic either to H_2 or to $\overline{H_2}$ depicted in Figure 2.

Conversely, let v be a vertex belonging to a rooted graph H of $\mathcal{F}_1$. If H belongs to $\{H_1, \overline{H_2}, H_4\}$ (resp. $\{H_2, \overline{H_4}\}$) then v is end-vertex in a symmetric wing in G (resp. $\overline{G}$). □

In [11] Olariu defined a graph G to be *quasi-brittle* if every induced subgraph H of G contains a special vertex, and proved that such a graph is perfectly orderable. Indeed, an attentive examination of his proof shows that he obtained the following more general result :

Theorem 1 *Let G be a graph and v be a special vertex in G. Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Note that such a result was known if v is semi-simplicial [7] or if v is a charming vertex [9].

Our purpose is to generalize these previous results and to obtain new families of perfectly orderable graphs by proving statements of the following type:

Assertion A *Let $\mathcal{F}$ be a family of rooted graphs. Let G be a graph and v be a $\mathcal{F}$-free vertex in G. Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

2 Definitions and notations

For terms not defined here, the reader is referred to [6]. All the graphs in this paper are simple and a graph is denoted $G = (V(G), E(G))$ (or simply $G = (V, E)$ if no confusion exists); its *complement* $\overline{G}$ is the graph $(V, \overline{E})$ where $\overline{E}=\{xy \mid x \in V, y \in V \text{and } xy \notin E\}$. For any subset A of V, the subgraph of G (resp. $\overline{G}$) induced by the set A will be denoted by $G[A]$ (resp. $\overline{G}[A]$).

For any vertex v in V, the *neighbourhood* $N_G(v)$ of v in G is defined as the set of all vertices adjacent to v in G. These vertices are called *neighbours* of v. When there is no possibility of confusion, the notation $N(v)$ will also be used to denote the neighbourhood of v. The set of vertices in G which are not adjacent to v is denoted by $M(v)$ (this set is equal to $N_{\overline{G}}(v)$, the neighbourhood of v in $\overline{G}$). For any subset W of $V \setminus \{v\}$ such that $W \cap N(v) \neq \emptyset$ and v is not adjacent to at least one vertex in W, we will say that v *sees* W *partially.* If $W \subseteq N(v)$, we will say that v *sees* W *completely.*

If $A \subseteq V$, then $G \setminus A$ will denote the graph induced by $V \setminus A$. The particular case where $A = \{x\}$ will be denoted $G - x$; also if H is a subgraph of G, we denote by $H + x$ the subgraph induced in G by $V(H) \cup \{x\}$ (if $x \notin V(H)$).

For any path P, the *length* of P is the number of its edges. If $V(P) = \{v_1, \ldots, v_k\}$ and $E(P) = \{v_i v_{i+1} \mid i \in \{1, \ldots, k-1\}\ \}$, P is also denoted by $[v_1, \ldots, v_k]$. The vertices v_1 and v_k are its *end-vertices* while any vertex v_i, with $1 < i < k$, is said to be an *internal* vertex. An induced path on k vertices will be denoted by P_k and an induced subgraph of G isomorphic to a P_k will be simply said to be *a P_k in G*. If u and v are vertices of a path P then $P[u, v]$ will denote the subpath of P whose end-vertices are u and v. For a P_4 with vertices a, b, c, d (in this order on the path), the notation will be slightly changed: it will be denoted $abcd$. The two internal vertices b and c will be called *midpoints* while the end-vertices a and d will be also referred as *endpoints*.

Similarly, a chordless cycle on k vertices is denoted by C_k or by $[v_1, \ldots, v_k, v_1]$ if its vertex set is $\{v_1, \ldots, v_k\}$ and its edge set is $\{v_i v_{i+1} \mid 1 \leq i \leq k-1\} \cup \{v_k v_1\}$.

A set $A \subseteq V$ of vertices is called a *module* if every vertex in $V \setminus A$ is either adjacent to all vertices in A, or none of them. A module in G is also a module in $\overline{G}$. A module is an *homogeneous set* if it is not a *trivial* one, *i.e.* not an empty set, a singleton or V itself. A graph with more than two vertices is *prime* if it only has trivial modules. Note that P_4 is the smallest prime graph and that any prime graph distinct from P_4 has at least five vertices.

Let $\mathcal{Z}$ be a set of graphs. We say that a graph G is *$\mathcal{Z}$-free* if no induced subgraph of G is isomorphic to a graph of $\mathcal{Z}$. A set $\mathcal{F}$ of graphs is *$\mathcal{Z}$-free* if every graph of $\mathcal{F}$ is $\mathcal{Z}$-free.

Let G and H be two graphs and let v be a vertex in G. The *χ-join* of H and G into v is the graph obtained from the union of H and $G - v$ by adding all the edges xy with $x \in V(H)$ and $y \in N_G(v)$; this graph is denoted by $\chi(H, G; v)$. Since an induced P_4 in $\chi(H, G; v)$ not contained in H and not in $G - v$ has exactly one vertex in H and the three other vertices in $G - v$, note that if $v_1 < v_2 < \cdots < v_n$ is a perfect order of G and if $w_1 < w_2 < \cdots < w_p$ is a perfect order of H, then $v_1 < \cdots < v_{i-1} < w_1 < w_2 < \cdots < w_p < v_{i+1} < \cdots < v_n$ is a perfect order of $\chi(H, G; v_i)$. Then we have

Lemma 3 *[3] Let G and H be two perfectly orderable graphs and let v be a vertex in G. Then $\chi(H, G; v)$ is perfectly orderable.*

3 Pleasant vertices

A vertex v in a graph G is said to be a *pleasant vertex* if v is $\mathcal{F}_0$-free, with $\mathcal{F}_0 = \{H_1, H_2, \overline{H_2}\}$ where these rooted graphs are described in Figure 2. Note that special vertices and nice vertices are pleasant vertices. We call *pleasant* any graph in which every induced subgraph has a pleasant vertex.

For any vertex v in a graph G let $C_1(v), \cdots, C_{q_v}(v)$ be the connected components of $\overline{G}[N(v)]$, and let $Q_1(v), \cdots, Q_{k_v}(v)$ be the connected components of $G[M(v)]$. For the sake of simplicity we will denote these components by $C_1, \cdots, C_q$ and by $Q_1, \cdots, Q_k$.

We give here a structural result that we will often use.

Proposition 1 *Let v be a pleasant vertex in a graph $G = (V, E)$. For every $i \in \{1, \cdots, q\}$ and every $j \in \{1, \cdots, k\}$ one of the following statement is true:*
1) every vertex in C_i is adjacent to every vertex in Q_j,
2) every vertex in C_i has no neighbour in Q_j,
3) there exists a non trivial part A_i of C_i such that for every vertex x in C_i and for every vertex y in Q_j, xy belongs to E if and only if x belongs to A_i,
4) there exists a non trivial part B_j of Q_j such that for every vertex x in C_i and for every vertex y in Q_j, xy belongs to E if and only if y belongs to B_j.

Proof. If $|C_i| = 1$ or $|Q_j| = 1$ then the result is clearly true. We suppose that $|C_i| \geq 2$, $|Q_j| \geq 2$ and that none of the conditions 1), 2) and 3) is satisfied. Then, there exist $x \in C_i$, a and $b \in Q_j$ such that $xa \in E$ and $xb \notin E$. Since Q_j is connected in G, we can suppose without loss of generality that $ab \in E$. Since C_i is connected in $\overline{G}$, let $y \in C_i$ distinct from x such that $xy \notin E$. Since the set of vertices $\{y, v, x, a, b\}$ induces no cycle C_5, no P_5 and no $\overline{P_5}$, then $ya \in E$ and $yb \notin E$.

Now we want to prove that any vertex h in Q_j is either adjacent to every vertex in C_i or adjacent to no vertex in C_i. A contrario, suppose that there exist h in Q_j, c and d in C_i such that $hc \in E$ and $hd \notin E$. Since C_i is connected in $\overline{G}$, we suppose without loss of generality that $cd \notin E$. Let k be a neighbour of h in Q_j. Since the set $\{k, h, c, v, d\}$ induces no C_5, no P_5 and no $\overline{P_5}$, then $kc \in E$ and $kd \notin E$. Since Q_j is connected, c is adjacent to every vertex in Q_j (and d is adjacent to no vertex in Q_j). But b belongs to Q_j and is adjacent to no vertex in C_i, so we obtain a contradiction. Thus, the statement 4) is true. $\square$

Corollary 1 *Let v be a pleasant vertex in a graph $G = (V, E)$.*

(i) Let C_i be a component of $\overline{G}[N(v)]$. Let a and b be two vertices in $M(v)$ such that $N(a) \cap C_i \neq \emptyset$, $N(b) \cap C_i \neq \emptyset$ and $N(a) \cap C_i \neq N(b) \cap C_i$. Then a and b belong to distinct components of $G[M(v)]$.

(ii) Let Q_j be a component of $G[M(v)]$. Let a and b be two vertices in $N(v)$ such that $N(a) \cap Q_j \neq \emptyset$, $N(b) \cap Q_j \neq \emptyset$ and $N(a) \cap Q_j \neq N(b) \cap Q_j$. Then a and b belong to distinct components of $\overline{G}[N(v)]$.

4 Acyclic orientation and obstructions

A perfect order of G can be seen as an acyclic orientation of the edges of G such that every induced P_4, $abcd$, is not an obstruction, that is $abcd$ cannot have simultaneously (a, b) and (d, c) as arcs. Our aim is the following: given a perfect order $<$ of $G - v$, how to orient the edges incident with v and how to modify the orientation of some edges in $G - v$ so that the oriented graph G has no circuit and no obstruction. A possibility to define a linear order $<'$ of G is the following: let us partition $V - v$ into L and R, and denote $N(v) \cap L$ by LH, $M(v) \cap L$ by LL, $N(v) \cap R$ by RH and $M(v) \cap R$ by RL; we do not modify the orientations in $G[L]$ and $G[R]$, every edge xy such that $x \in L$ and $y \in R$ is oriented such that $x <' y$, every edge xv with $x \in L$ is oriented such that $x <' v$ and, at last, every edge yv such that $y \in R$ is oriented such that $v <' y$ (see Figure 5). Now, our problem is to define the sets LH, LL, RH and RL in a suitable way to avoid obstructions.

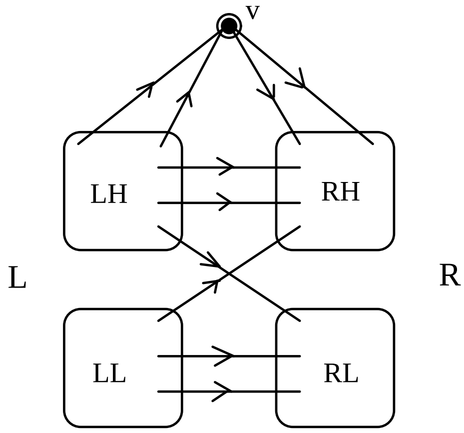

Fig. 5

We note that, to obtain their result, Hoàng and al. [9] set $LH = N(v)$, $LL = \emptyset$, $RH = \emptyset$ and $RL = M(v)$.

We consider a pleasant vertex v of the graph G, and we suppose $G - v$ perfectly orderable. Let $<$ be a perfect order on $G - v$. Following a similar idea as used by Olariu in [11], a component C_i of $\overline{G}[N(v)]$ is called *impure* if a vertex a in $M(v)$ and vertices b, d in C_i exist such that $abvd$ is a P_4 and $a < b$. A component C_i is called *pseudo-pure* if there are vertices a, b in $M(v)$ and a vertex c in C_i such that $abcv$ is a P_4 and $a < b$. A component which is neither impure nor pseudo-pure is referred to as *pure*.

Lemma 4 *No pseudo-pure component is impure.*

Proof. Assume that C_i is both pseudo-pure and impure : let a, b, x in $M(v)$ and c, y, z in C_i be such that $abcv$ and $xyvz$ are P_4 with $a < b$ and $x < y$. Since $ac \notin E$ and $bc \in E$, the vertex b sees C_i completely (by Proposition 1). As $bz \in E$, $xy \in E$ and $xz \notin E$, the component Q_j containing x is different from the component containing a and b (by Corollary 1). Then $abyx$ is an obstruction in $(G - v; <)$, a contradiction.

We can now define the sets LH, RH, LL and RL : LH contains all the pseudo-pure components and, perhaps, some pure components ; RH contains all the impure components and the remaining pure components ; LL is the union of some components Q_j and RL is the union of the other components of $G[M(v)]$ (LL or RL can be empty). By construction, the oriented graph $(G; <')$ presents no circuit.

Claim 1 $(G; <')$ *presents no obstruction containing* v.

Proof. If $abcv$ is an obstruction in $(G; <')$, then a and b belong to the same component Q_j and $a < b$. Thus, c belongs to a pseudo-pure component and $v <' c$, a contradiction. If $abvd$ is an obstruction in $(G; <')$, then the component C_i containing b and d is contained in LH $(d <' v)$ and a is in LL $(a <' b)$. Thus $a < b$ and C_i is impure, a contradiction. $\square$

We are going to observe now the possible obstructions. Let $abcd$ be an obstruction in $(G; <')$. Without loss of generality, we can suppose that $a \in L, b \in R$ and $b < a$. Since there is no edge between LL and RL, we notice that a or b is a neighbour of the vertex v.

Proposition 2 *If $abcd$ is an obstruction in $(G; <')$, then one of the following situations depicted in Figure 6 is realised :*
Situation 1 : $a \in LH$ and $b, c, d \in Q_j \subseteq RL$.
Situation 2 : $a \in LL, b, c \in RH$ and $d \in RL$.
Situation 3 : $a \in LL$ and $b, c, d \in RH$.
Situation 4 : $a \in LL, b, d \in RH$ and $c \in RL$.
Situation 5 : $a, d \in LL$ and $c, b \in RH$.
Situation 6 : $a, d \in LL, c \in LH$ and $b \in RH$.

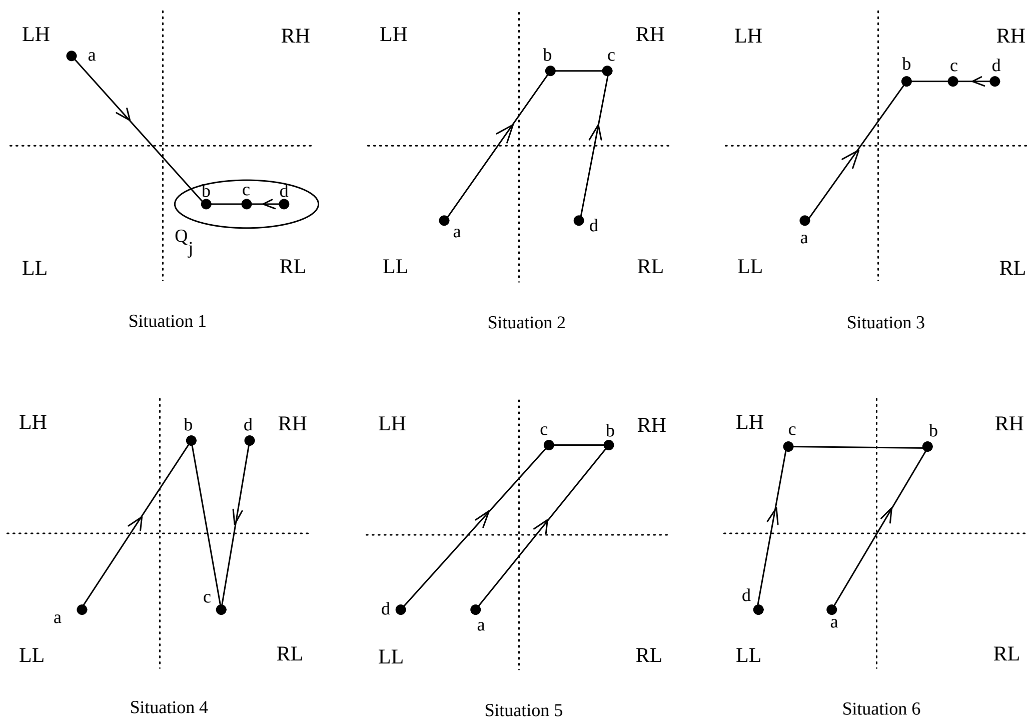

Fig. 6

Proof. Since $d <' c$, we notice that $c \notin L$ while $d \in R$. We distinguish three cases :
Case A : $c, d \in R$;
Case B : $d \in L$ and $c \in R$;
Case C : $c, d \in L$.

Case A. $c, d \in R$, then $d < c$.

Case A1. We suppose that $a, b \in N(v)$. Let C_i be the component (pure or pseudo pure) containing a and let C_j be the component (pure or impure) containing b. Since $ad \notin E$ and $bd \notin E$, d belongs to $M(v)$. If $c \in N(v)$ then c belongs to C_i and this component is impure (because $d < c$), a contradiction. Thus, c and d belong to $M(v)$. But now, we consider the P_4 $dcbv$ and we see that C_j is pseudo-pure, which gives also a contradiction. Then, Case A1 does not occur.

Case A2. We suppose that $a \in N(v)$ and $b \notin N(v)$. As $ac, ad \notin E$, b, c and d belong to the same component Q_j. This is the first situation.

Case A3. We suppose that $a \notin N(v)$ and $b \in N(v)$. As the component containing b is not pseudo-pure, one of the vertices c or d is in $N(v)$. We obtain three other possibilities: Situations 2, 3 and 4.

Case B. $d \in L$ and $c \in R$.

Case B1. We suppose that $a \in N(v)$. Since ac, $bd \notin E$ and $cd \in E$, c belongs to $M(v)$ and d belongs to $N(v)$. Since ad, $bd \notin E$, a and d are in the same component C_i, b and c are in the same component Q_j. By Proposition 1, $abcd$ cannot be a P_4; this is a contradiction. Then, Case B1 cannot occur.

Case B2. We suppose that $a \notin N(v)$ and $b \in N(v)$. Since $bd \notin E$ and $cd \in E$, we have $d \notin N(v)$ and $c \in N(v)$. We obtain the fifth situation.

Case C. $c, d \in L$, then $d < c$.

Case C1. We suppose that $a, b \in N(v)$. As $bd \notin E$, necessarily $d \notin N(v)$. If $c \in N(v)$, then c belongs to the component C_i containing a, and C_i is impure ; this is a contradiction. If $c \notin N(v)$, we consider the graph induced by $\{d, c, b\}$ and we see that the component $C_{i'}$ containing b is pseudo-pure. This is also a contradiction ; then Case C1 is impossible.

Case C2. We suppose that $a \in N(v)$ and $b \notin N(v)$. Necessarily, a and c are in the same component C_i. If $d \in N(v)$, then d is in the component C_i and $bavd$ is a P_4 where $b < a$. Then C_i is impure, which is impossible. If $d \notin N(v)$, then $dcva$ is a P_4 and $d < c$. So the component C_i is impure. Thus, Case C2 is impossible.

Case C3. We suppose that $a \notin N(v)$ and $b \in N(v)$. If $c \notin N(v)$, then $dcbv$ is a P_4 where $d < c$, and the component containing b is pseudo-pure. So $c \in N(v)$, and we obtain the last situation. □

From the previous proposition, we deduce immediately the two following results (we shall see later that we can improve Corollary 3).

Corollary 2 *Let v be a pleasant vertex of a graph G. If v is endpoint of no P_5 then G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Proof. We choose $LL = \emptyset$ and $RL = M(v)$: none of situations 2 to 6 appears. Since v is endpoint of no P_5, Situation 1 cannot be realised. □

Corollary 3 *Let v be a pleasant vertex of a graph G. If v is $\{H_5, H_6\}$-free (H_5 and H_6 are described in Figure 7), then G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

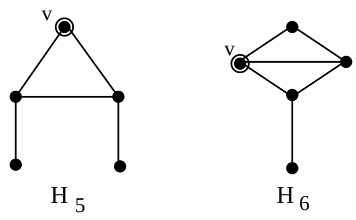

Fig. 7

Proof. We choose $LL = M(v)$ and $RL = \emptyset$; the situations 1, 2, 4 cannot appear. Since v is a vertex $\{H_5, H_6\}$-free , none of the situations 3, 5, and 6 is realised. ∎

5 Main results

In [5] we proved the two following results which generalize Theorem 1:

Theorem 2 *Let G be a graph and v be a vertex $\mathcal{F}_2$-free, where $\mathcal{F}_2$ is the family of rooted graphs $\{H_1, H_2, \overline{H_2}, \overline{H_4}\}$ (see Figures 2 and 4). Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

Theorem 3 *Let G be a graph and v be a $\mathcal{F}_3$-free vertex, where $\mathcal{F}_3$ is the family of rooted graphs $\{H_1, H_2, \overline{H_2}, H_7, H_8\}$ (see Figures 2 and 9). Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

An other way to generalize Theorem 1 is to consider the family $\{H_1, H_2, \overline{H_2}, H_4\}$. We show below a no perfectly orderable graph G such that $G - v$ is perfectly orderable and v is $\{H_1, H_2, \overline{H_2}, H_4\}$-free.

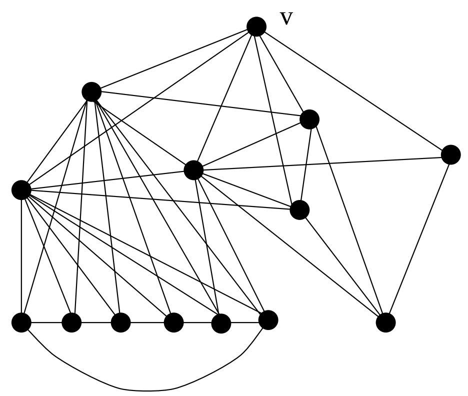

Fig. 8

We note that the vertex v is H_8-rooted. In this report, we consider the family $\{H_1, H_2, \overline{H_2}, H_8\}$ and prove:

Theorem 4 *Let G be a graph and v be a $\mathcal{F}_4$-free vertex, where $\mathcal{F}_4$ is the family of rooted graphs $\{H_1, H_2, \overline{H_2}, H_8\}$ (see Figures 2 and 9). Then, G is perfectly orderable if and only if $G - v$ is perfectly orderable.*

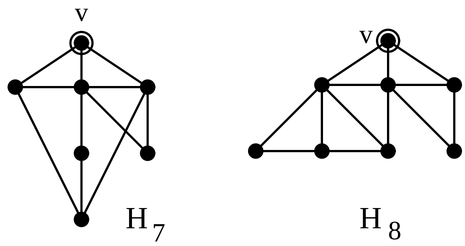

Fig. 9

The proof of this theorem is the topic of the next section. We give now some useful notations and results. Let $<$ be a perfect order on $G - v$. Since v is a pleasant vertex of G, by Lemma 4, no pseudo-pure component is impure. Then, let us consider the linear order $<'$ on G defined in Section 4, such that LH contains no pure component (that is LH is the union of the pseudo-pure components) and LL is empty (that is $RL = M(v)$). The oriented graph $(G; <')$ presents no circuit and, by Claim 1, no obstruction containing v. Let $A = abcd$ be an obstruction in $(G; <')$. Since $LL = \emptyset$, by Proposition 2, Situation 1 is the only one possibility. We suppose without loss of generality that a belongs to LH and b, c and d belong to RL. We denote by $C(A)$ the pseudo-pure component of $\overline{G}[N(v)]$ containing a and by $Q(A)$ the component of $G[M(v)]$ containing b, c and d. Note that, by Proposition 1, for every $x \in C(A)$ we have $bx \in E$, $cx \notin E$ and $dx \notin E$. Moreover, $a <' b$, $b < a$ and $d < c$.

Claim 2 *There exist s and t in $Q(A)$ such that $t < s$ and for all x in $C(A)$, $sx \in E$ and $tx \notin E$.*

Proof. Since $C(A) \subseteq LH$, $C(A)$ is a pseudo-pure component. Then, there exist t and s in a component Q_j such that $t < s$, $st \in E$ and for all x in $C(A)$, $sx \in E$ and $tx \notin E$. If $Q_j \neq Q(A)$ then $tsab$ is an obstruction in $(G - v; <)$, a contradiction. Thus, $Q_j = Q(A)$. □

Let $\mathcal{A}$ be the set of obstructions in $(G; <')$ and let $\mathcal{C}(\mathcal{A})$ be the set of vertices x in R such that there exist an obstruction $A = abcd$ in $\mathcal{A}$ and an oriented path, $P[x, b]$, from x to b, in $(G - v; <)$.
Remark that there is no edge xy such that $x \in \mathcal{C}(\mathcal{A})$, $y \in R \setminus \mathcal{C}(\mathcal{A})$ and $y < x$. Otherwise, y belongs to $\mathcal{C}(\mathcal{A})$, a contradiction.

Let V_1 be the set of all vertices $v_1 \in \mathcal{C}(\mathcal{A})$ such that two vertices v_2 and v_3 exist satisfying $v_1 v v_2 v_3 \equiv P_4$ and $v_3 < v_2$.
Note that such a vertex v_1 necessarily belongs to an impure component.

In [5], to prove Theorem 3, we applied the rules given in Section 4 to the partition $\{L' = L \cup \mathcal{C}(\mathcal{A}), R' = R \setminus \mathcal{C}(\mathcal{A})\}$. This partition is not appropriate when v is not H_7-free: obstructions of type $v_1vv_2v_3$ where $v_1 \in V_1$ can appear. An idea is to refine this partition: in Section 6, we define a new partition $\{L' = L \cup \mathcal{V}, R' = R \setminus \mathcal{V}\}$ such that $\mathcal{V} \subset \mathcal{C}(\mathcal{A})$ and $V_1 \subset R'$. Before, we need some more definitions and results.
For every vertex $x \in \mathcal{C}(\mathcal{A})$ we shall say that a triplet $(A = abcd, s, t)$ is *linked* with x if $A \in \mathcal{A}$, $s, t \in Q(A)$, $ast \equiv P_3$, $t < s$ and if an oriented path from x to b exists in $(G - v, <)$.

For every $v_1 \in V_1$, let $\mathcal{C}(v_1)$ be the set of vertices $x \in \mathcal{C}(\mathcal{A})$ such that an oriented path $P[v_1, x]$ exists in $(G - v, <)$, and let $\mathcal{C}(V_1)$ be the union of all sets $\mathcal{C}(v_1)$.

Claim 3 *Let v_1, v_2, v_3 be vertices such that $v_1 \in V_1$, $v_1vv_2v_3 \equiv P_4$ and $v_3 < v_2$. For every triplet $(A = abcd, s, t)$ linked with v_1, we have: $v_3 \notin Q(A)$, $v_1b, v_2b \notin E$, $v_1t, v_2t, av_3 \in E$, $t < v_1$, $t < v_2$, $v_1 < a$, $a < v_2$, and $a < v_3$.*

Proof. We call C the impure component containing vertices v_1 and v_2. Since (A, s, t) is linked with v_1, there exists in $(G - v, <)$ an oriented path $P[v_1, b]$ from v_1 to b. The component $C(A)$ is pseudo-pure, then, from Lemma 4, $C(A)$ and C are different components. Then we have av_1, $av_2 \in E$. Since $b < a$, we have $v_1 < a$ (otherwise, $P[v_1, b]av_1$ is a circuit in $(G - c, <)$). Since $v_1av_2v_3$ is not an obstruction, vertices v_3 and a are adjacent. If $v_1b \in E$, then $v_1 < b$ (otherwise, $P[v_1b]v_1$ is a circuit). By Corollary 1, v_3 does not belong to $Q(A)$. Since $v_1bv_2v_3$ is not an obstruction, we have $bv_2 \notin E$. By Proposition 1, v_1 sees $Q(A)$ completely and v_2 has no neighbour in $Q(A)$. Now, the subgraph of G induced by $\{v, a, v_1, v_2, v_3, b, c, d\}$ is isomorphic to the pointed graph H_8, a contradiction. Thus, vertices v_1 and b are nonadjacent.
Since v_1ast is not an obstruction in $(G - v, <)$, $v_1s \in E$ or $v_1t \in E$. Then, by Proposition 1, v_1 and v_2 have the same neigbourhood in $Q(A)$ (then $v_2b \notin E$). Thus, by Corollary 1, v_3 does not belong to $Q(A)$.
If v_1 and t are nonadjacent, then we have $v_1s \in E$, $v_2t \notin E$, $v_2s \in E$. Then, tsv_2v_3 is an obstruction in $(G - v, <)$, a contradiction. Thus, vertices v_1 and v_2 are adjacent to t.
Since $tsav_3$ is not an obstruction, we have $a < v_3$. Then we have $a < v_2$ (otherwise, av_3v_2a is a circuit). Finally, we have $t < v_1$ (otherwise, $v_1tv_2v_3$ is an obstruction), and $t < v_2$ (otherwise, tv_1av_2t is a circuit). $\square$

Lemma 5 *If $V_1 \neq \emptyset$, then there exists exactly one component in $M(v)$ (denoted Q_0) containing vertices b, c, d of every obstruction $A = abcd \in \mathcal{A}$ such that $b \in \mathcal{C}(V_1)$. Moreover, there exists an impure component, denoted C_0, containing the set V_1.*

Proof. Consider two obstructions $A = abcd$ and $A' = a'b'c'd'$ in $\mathcal{A}$ such that $b, b' \in \mathcal{C}(V_1)$, and four vertices s, t, s', t' such that $ast \equiv a's't' \equiv P_3$, $t < s$ and $t' < s'$. We assume that the component $Q(A')$ is distinct from $Q(A)$.
Vertices a and a' belong to two different pseudo-pure components; otherwise, we have $ab' \in E$, $b' < a$ and $b'ast$ is an obstruction. Then, a and a' are adjacent, and we can suppose, without loss of generality, that $a < a'$. Vertices a' and b are nonadjacent;

otherwise, either $a' < b$ and $a'baa'$ is a circuit, or $b < a'$ and $ba's't'$ is an obstruction. Since $baa'b'$ and $tsab'$ are not obstructions, we have $ab' \in E$ and $a < b'$. Since $t's'a'a$, $t's'ab$ and $stat'$ are not obstructions, we have $at' \in E$ and $a < t'$. Let $v_1 \in V_1$ be a vertex such that $b \in \mathcal{C}(v_1)$ (such a vertex exists because $b \in \mathcal{C}(V_1)$). The triplet (A, s, t) is linked with v_1 then, by Claim 3, we have $v_1 < a$. Now, since $v_1 < a < b'$, the triplet (A', s', t') is linked with v_1. According to Claim 3, we have $t' < v_1$, then $t'v_1at'$ is a circuit, a contradiction. Thus, we have $Q(A') = Q(A)$.
Let us consider two vertices in V_1, v_1 and v_1', belonging to different impure components, respectively denoted C_1 and C_1'. According to the definition of V_1, vertices v_2 and v_3 (resp. v_2' and v_3') exist such that $v_1vv_2v_3 \equiv P_4$ and $v_3 < v_2$ (resp. $v_1'vv_2'v_3' \equiv P_4$ and $v_3' < v_2'$). Since v_1, $v_2 \in C_1$ and v_1', $v_2' \in C_1'$, v_1v_1', v_1v_2', v_2v_1' and v_2v_2' are edges in G. We can suppose, without loss of generality, that $v_1 < v_1'$. Let $(A = abcd, s, t)$ be linked with v_1': there exist an oriented path $P[v_1', b]$ from v_1' to b. The path $v_1P[v_1', b]$ is an oriented path from v_1 to b, then (A, s, t) is linked with v_1. According to Claim 3, we have $v_1 < a$, $v_1' < a$, $a < v_2$, $a < v_2'$, $av_3 \in E$, $av_3' \in E$, $a < v_3$ and $a < v_3'$. Moreover, v_3 and v_3' do not belong to Q_0. Since $v_1v_1'v_2v_3$ is not an obstruction, we have $v_3v_1' \in E$. Hence, by Corollary 1 v_3 and v_3' belong to different components in $G[M(v)]$, and $v_3v_3' \notin E$. Since $v_1av_2'v_1$ and $v_1'av_2v_1'$ are not circuits, we have $v_1 < v_2'$ and $v_1' < v_2$. Since $v_1v_2'v_2v_3$ is not an obstruction, vertices v_3 and v_2' are adjacent. Since $v_1'v_3v_2'v_3'$ is not an obstruction, we have $v_3 < v_1'$. So, $av_3v_1'a$ is a circuit in $(G - v, <)$, a contradiction. □

Claim 4 *If a vertex $x \in Q_0$ is adjacent to a vertex in C_0, then x sees C_0 completely. Moreover, for every vertices v_1, v_2, v_3 such that $v_1 \in V_1$, $v_1vv_2v_3 \equiv P_4$ and $v_3 < v_2$, we have $x < v_1$ and $x < v_2$. Consequently, every vertex in $\mathcal{C}(V_1) \cap Q_0$ has no neighbour in C_0 (in particular, if (A, s, t) is linked with a vertex in V_1, then $t \notin \mathcal{C}(V_1)$).*

Proof. Let us choose such vertices v_1, v_2, v_3, and a triplet $(A = abcd, s, t)$ linked with v_1. By Lemma 5, vertices t and b belong to Q_0 and v_1 belongs to C_0. By Claim 3, we have $v_1b \notin E$, $t < v_1$, $v_1 < a$, $a < v_2$ and $v_3 \notin Q_0$. Since t is adjacent to v_1 and b is non-adjacent to v_1, any vertex x in Q_0 is adjacent to every vertice in C_0 or none. Therefore, if $x \in Q_0$ is adjacent to v_1 and v_2, then we have $x < v_1$ (otherwise, $v_1xv_2v_3$ is an obstruction in $(G - v, <)$) and $x < v_2$ (otherwise, $v_2xv_1av_2$ is a circuit).
Let y be a vertex in $\mathcal{C}(V_1) \cap Q_0$: there exist $v_1' \in V_1$ and an oriented path $P[v_1', y]$ from v_1' to y. If y is adjacent to v_1', then $y < v_1'$ and $yP[v_1', y]$ is a circuit, a contradiction. □

6 Proof of Theorem 4

We denote $\mathcal{U}_M$ the set
$\{x \in M(v) \cap \mathcal{C}(V_1) | \exists (A = abcd, s, t)$ linked with x such that $ax \in E\}$.

Claim 5 *For every vertex $x \in \mathcal{U}_M$, if $(A = abcd, s, t)$ is linked with x and $ax \in E$, then $x < a$, t sees C_0 completely, $tx \in E$ and $t < x$.*

Proof. Let x be a vertex in $\mathcal{U}_M$ and $(A = abcd, s, t)$ be a triplet linked with x such that $ax \in E$. Since $aP[x, b]a$ is not a circuit, we have $x < a$. We consider vertices $v_1 \in V_1$,

v_2 and v_3 such that $x \in \mathcal{C}(v_1)$, $v_1v_3, v_1v_2 \notin E$, $v_2v_3 \in E$ and $v_3 < v_2$ (by Lemma 5, $v_1, v_2 \in C_0$). Since there exist oriented paths $P[v_1, x]$ from v_1 to x and $P[x, b]$ from x to b, the triplet (A, s, t) is linked with v_1. According to Claim 3, t sees C_0 completely and $t < v_1$. According to Claim 4, the vertex x, which belongs to $\mathcal{C}(V_1)$, has no neighbour in C_0. Since tv_1ax is not an obstruction in $(G - v, <)$, t is adjacent to x. By Claim 3, v_3 does not belong to Q_0 and v_3v_2tx is an induced P_4. Then we have $t < x$, otherwise v_3v_2tx is an obstruction in $(G - v, <)$. □

Claim 6 $\mathcal{U}_M \subset Q_0$

Proof. Let x be a vertex in $\mathcal{U}_M$ and $(A = abcd, s, t)$ be linked with x such that $ax \in E$. By Claim 5, t is adjacent to x, so x and t belong to the same component in $M(v)$. By Lemma 5, this component is Q_0. □

Claim 7 *For every vertices $y \in (M(v) \cap \mathcal{C}(V_1)) \setminus \mathcal{U}_M$ and $x \in \mathcal{U}_M$, if x is adjacent to y, then $x < y$.*

Proof. Let y be a vertex in $(M(v) \cap \mathcal{C}(V_1)) \setminus \mathcal{U}_M$ and x be a vertex in $\mathcal{U}_M$, adjacent to y. Vertices x and y belong to the same component in $M(v)$, Q_0 (by Claim 6). We consider a vertex $v_1 \in V_1$ and a triplet $(A = abcd, s, t)$ such that $y \in \mathcal{C}(v_1)$, (A, s, t) is linked with x and $ax \in E$. By Claim 5, we have $x < a$. Since x, y belong to $\mathcal{C}(V_1)$, by Claim 4, v_1 is nonadjacent to vertices x and y.
If $y < x$, then $yP[x, b]$ is an oriented path from y to b, and (A, s, t) is linked with y. Then, y is nonadjacent to a (otherwise, $y \in \mathcal{U}_M$), and $yxav_1$ is an obstruction in $(G - v, <)$, a contradiction. □

Claim 8 *Let x and y be vertices such that $y \in (N(v) \cap \mathcal{C}(V_1))$, $x \in \mathcal{U}_M$, $xy \in E$ and $y < x$. If $(A = abcd, s, t)$ is linked with x and $xa \in E$, then $ty \in E$ and $t < y$.*

Proof. By Claim 5, we have $yt \in E$ (otherwise, y would belong to a pseudo-pure component and $y \notin \mathcal{C}(V_1)$).
We consider $v_1 \in V_1$ such that $y \in \mathcal{C}(v_1)$. The triplet (A, s, t) is linked with v_1 (there exists an oriented path from v_1 to y and $y < x$) then, by Claim 3, we have $t < v_1$. Since $(G - v, <)$ has no circuit, we have $t < y$. □

Lemma 6 *Let x be a vertex in $\mathcal{U}_M$ and $(A = abcd, s, t)$ be a triplet linked with x such that $ax \in E$. If $y'vyx \equiv P_4$, then y sees Q_0 completely, y' has no neighbour in Q_0, $y < x$ and $y < t$. Moreover, y does not belong to $\mathcal{C}(V_1)$.*

Proof. We consider $v_1 \in V_1$ such that $x \in \mathcal{C}(v_1)$, and vertices v_2, v_3 such that $v_1v_3, v_1v_2 \notin E$, $v_2v_3 \in E$ and $v_3 < v_2$ (by Lemma 5, vertices v_1 and v_2 belong to C_0). Since oriented paths $P[v_1, x]$ and $P[x, b]$ exist, the triplet (A, s, t) is linked with v_1. By Claim 5, we have $tx \in E$ and $t < x$. Moreover, according to Claim 3, we have $tv_1 \in E$, $t < v_1$ and $v_1 < a$.
Let y and y' be vertices such that $y'vyx \equiv P_4$; y and y' belong to the same component C_i

in $\overline{G}[N(v)]$. Since y is adjacent to $x \in Q_0$, while y' is non-adjacent to x, by Proposition 1, y sees Q_0 completely and y' has no neighbour in Q_0. The component containing a and the component containing v_1 and v_2 are completely seen by some vertices in Q_0, so this components are distinct from C_i. Hence, we successively have: $a < y'$ (otherwise, $txay'$ is an obstruction); $v_1 < y'$ (otherwise, $ay'v_1a$ is a circuit); $y'v_3 \in E$ (otherwise, $v_1y'v_2v_3$ is an obstruction); $y' < v_3$ (otherwise, $tv_1y'v_3$ is an obstruction); $yv_3 \in E$ (otherwise, $G[v, y, a, y', d, c, b, v_3] \equiv H_8$); $y < t$ (otherwise, $y'v_3yt$ is an obstruction); $y < x$ (otherwise, $ytxy$ is a circuit).
Now, since $y < t$, by Claim 8, y does not belong to $\mathcal{C}(V_1)$. $\square$

Claim 9 *Let x, y be vertices such that $x \in \mathcal{U}_M$, $y \in N(v) \cap \mathcal{C}(V_1)$, $yx \in E$ and $y < x$. We note C_i the component containing y in $\overline{G}[N(v)]$. Let $(A = abcd, s, t)$ be linked with x such that $ax \in E$. If y' is a vertex in $C_i \cap \mathcal{C}(V_1)$ such that there exists, in C_i, an oriented path $P[y', y]$ from y' to y, then we have $y'x, y't \in E$, $y' < x$ and $t < y'$.*

Proof. We choose $v_1 \in V_1$ such that $y' \in \mathcal{C}(v_1)$. Clearly, vertices y and x belong to $\mathcal{C}(v_1)$, and the triplet $(A = abcd, s, t)$ is linked with vertices y, y' and v_1 (in particular, there exists an oriented path $P[y', b]$ from y' to b). According to Claim 3 , Claim 5 and Claim 8, we have: $tx \in E$, $t < x$, $tv_1 \in E$, $t < v_1$, $ty \in E$ and $t < y$. Since $P[y', b]ay'$ is not a circuit, we have $y' < a$, and since $y'axt$ is not an obstruction, y' is adjacent either to x or to t. Then, vertices y and y' have the same neighbourhood in Q_0, and $y'x, y't \in E$. Since $tv_1y't$ and $P[y', y]xy'$ are not circuits, we have $t < y'$ and $y' < x$. $\square$

Let us consider the set $\mathcal{Y}$ of all vertices in $RH \cap \mathcal{C}(V_1)$ which are adjacent to any vertex x in $\mathcal{U}_M$ such that $y < x$. By Claim 4, $\mathcal{Y}$ does not intersect C_0.

Remark 1 *If a vertex y' in $\mathcal{C}(V_1)$ is in the same component of $\overline{G}[N(v)]$ as a vertex $y \in \mathcal{Y}$, and y' is adjacent to y, then, by Claim 9, $y' \in \mathcal{Y}$ or $y < y'$.*

We consider now two subsets of $\mathcal{Y}$: $\mathcal{Y}'$ is the set of all vertices $y \in \mathcal{Y}$ such that y is not adjacent to a vertex x in $\mathcal{U}_M$ and an oriented path $P[y, x]$ exists; $\mathcal{Y}''$ is the set of all vertices $y \in \mathcal{Y}$ such that vertices x, x_1, x_2 exist and: $x \in \mathcal{U}_M$, $x_1 \in N(v)$ and x_1 does not belong to the component of $\overline{G}[N(v)]$ containing y, $x_2 \in M(v)$, $y < x$, $xyx_1x_2 \equiv P_4$ and $x_2 < x_1$.

We now define a linear order $<''$ on the graph G: let $\mathcal{U}_N$ be the set of all vertices y' such that there exist a vertex $y \in \mathcal{Y}' \cup \mathcal{Y}''$ and, in the component of $\overline{G}[N(v)]$ containing y, an oriented path from y' to y; $\mathcal{U} = \mathcal{U}_M \cup \mathcal{U}_N$, $\mathcal{R} = \mathcal{C}(V_1) \setminus \mathcal{U}$ and $\mathcal{V} = \mathcal{C}(\mathcal{A}) \setminus \mathcal{R}$. Now, set $L' = L \cup \mathcal{V}$ and $R' = R \setminus \mathcal{V}$. We apply to this new partition $\{L', R'\}$ of $V \setminus \{v\}$ the rules given in Section 4 to obtain from $(G - v; <)$ a new oriented graph $(G, <'')$. We can remark that $\mathcal{U}_N \subset \mathcal{Y}$ (by Remark 1) and $C_0 \subset R'$.

Claim 10 *If $x <'' y$ and $y < x$ then, one of the following propositions is true:*

1. *x belongs to a pseudo-pure component and $y \in R'$ (and $x <' y$);*

2. $x \in \mathcal{U}_N$ *and* $y \in \mathcal{R}$*;*

3. $x \in \mathcal{U}_M$ *and* $y \in \mathcal{Y} \cap \mathcal{R}$*.*

Proof. Let x and y be vertices such that $x <'' y$ and $y < x$. Clearly, we have $x \in L'$ and $y \in R'$. If x belongs to no pseudo-pure component, then x belongs to $\mathcal{V} \subset \mathcal{C}(\mathcal{A})$. Since $y < x$ and $y \in R'$, y belongs to $\mathcal{C}(\mathcal{A}) \setminus \mathcal{V}$ (because $x \in \mathcal{C}(\mathcal{A})$ and $\mathcal{V} \subset L'$). Since $\mathcal{V} = \mathcal{C}(\mathcal{A}) \setminus \mathcal{R}$, y belongs to $\mathcal{R}$, and then y belongs to $\mathcal{C}(V_1)$. Since $y < x$, x belongs to $\mathcal{C}(V_1)$, and then x belongs to $\mathcal{U}$.
According to Claim 7, if $x \in \mathcal{U}_M$, then $y \in N(v)$ and y belongs to $\mathcal{Y}$. ∎

Claim 11 *There is no obtruction* $a_1va_2a_3$ *in* $(G, <'')$*.*

Proof. We consider such an obstruction. If $a_3 < a_2$, then a_1 belongs to an impure component. Since $a_1 <'' v$, we have $a_1 \in L'$ and $a_1 \in \mathcal{C}(\mathcal{A})$. By the definition of V_1, a_1 belongs to $V_1 \subset R'$, a contradiction.
If $a_2 < a_3$ then, by Claim 10, $a_3 \in \mathcal{U}_{\mathcal{M}}$ and $a_2 \in \mathcal{Y} \cap \mathcal{R}$ (so, $a_2 \in \mathcal{C}(V_1)$). Since $a_1va_2a_3 \equiv P_4$, by Lemma 6, a_2 does not belong to $\mathcal{C}(V_1)$, a contradiction. ∎

Claim 12 *There is no obstruction* $va_1a_2a_3$ *in* $(G, <'')$*.*

Proof. We consider such an obstruction. Since a_2 and a_3 belong to $M(v)$, by Claim 10, $a_3 < a_2$. Thus, a_1 belongs to a pseudo-pure component. So, we have $a_1 <'' v$, and $va_1a_2a_3$ is not an obstruction in $(G, <'')$, a contradiction. ∎
The proof of Theorem 4 ends with the following result:

Claim 13 *There is no obstruction* $y_1y_2w_1w_2$ *in* $(G - v, <'')$*.*

Proof. We consider such an obstruction. We suppose, without loss of generality, that $y_2 < y_1$ (so, $y_1 \in L'$ and $y_2 \in R'$).

Case 1. $y_1 \in M(v)$.
According to Claim 10, we have $y_1 \in \mathcal{U}_M$ and $y_2 \in \mathcal{Y} \cap \mathcal{R}$.
• w_2 belongs to $M(v)$. Otherwise, w_2 belongs to $N(v)$ and $w_2vy_2y_1$ is a P_4. Then, by lemma 6, $y_2 \notin \mathcal{C}(V_1)$, a contradiction.
• w_1 belongs to $N(v)$. Otherwise, w_1 belongs to $M(v)$ and, by Claim 10, $w_2 < w_1$. Then, y_2 belongs to a pseudo-pure component, which contradicts $y_2 \in R'$.
• We have $w_2 < w_1$. Otherwise, according to Claim 10, w_2 belongs to $\mathcal{U}_M$ and w_1 belongs to $\mathcal{Y} \cap \mathcal{R}$. If $y_2 < w_1$ (resp. $w_1 < y_2$), then $y_2 \in \mathcal{Y}'$ (resp. $w_1 \in \mathcal{Y}'$), so $y_2 \in \mathcal{V} \subset L'$ (resp. $w_1 \in \mathcal{V}$), a contradiction.
• Vertices w_1 and y_2 belong to the same component of $\overline{G}[N(v)]$. Otherwise, y_2 belongs to $\mathcal{Y}'' \subset L'$, a contradiction.
Since y_2 is adjacent to $y_1 \in Q_0$ while w_1 is not adjacent to y_1, y_2 sees Q_0 completely and w_1 has no neighbour in Q_0. Then, Q_0 does not contain w_2. So, for every vertex z in Q_0, we have $y_2 < z$ (otherwise, $zy_2w_1w_2$ is an obstruction in $(G - v, <)$) which contradicts Claim 8.

Henceforth, we suppose that $y_1 \in N(v)$. In the same way, if $w_1 < w_2$, we suppose that $w_2 \in N(v)$.

Case 2. y_1 belongs to a pseudo-pure component.
Then we have $y_1 <' y_2$.
• One of the vertices w_1 and w_2 belongs to $N(v)$. Otherwise, vertices y_1, w_1 and w_2 belong to the same pseudo-pure component and $w_2 <' w_1$. The obstruction $y_1y_2w_1w_2$ is an obstruction in $(G - v, <')$ which does not satisfy Situation 1, a contradiction.
• w_2 belongs to $M(v)$. Otherwise, w_2 belongs to the pseudo-pure component containing y_1. Since $y_2 \in R'$ and $w_2y_2 \notin E$, $y_2 \in M(v)$ and y_2 belongs to the component containing w_1. Then, $vy_1y_2w_1w_2 \equiv C_5$, a contradiction.
• w_1 belongs to $N(v)$. Otherwise, w_1 and w_2 belong to the same component of $G[M(v)]$, and by Claim 10, $w_2 < w_1$ (and $w_2 <' w_1$). Then, $y_1y_2w_1w_2 \in \mathcal{A}$ and $y_2 \in \mathcal{V}$, a contradiction.
Then, w_1 and y_1 belong to the same pseudo-pure component and $w_1 < w_2$ (otherwise, this component is impure). According to Claim 10, we have $w_2 \in \mathcal{U}_M$ and $w_1 \in \mathcal{Y}$. So, $w_1 \in R$, a contradiction (w_1 belongs to a pseudo-pure component).
Henceforth, we suppose that $y_1 \in \mathcal{U}_N$ and $y_2 \in \mathcal{R}$. In the same way, if $w_1 < w_2$, we suppose that $w_2 \in \mathcal{U}_N$ and $w_1 \in \mathcal{R}$.

Case 3. $w_1 < w_2$.
Vertices w_2 and y_1 belong to the same component in $\overline{G}[N(v)]$. Since $y_2 < y_1$ and $y_2w_2 \notin E$, the vertex y_2 belongs to $M(v)$ (otherwise, y_2 belongs to $\mathcal{U}_N$). Similarly, w_1 belongs $M(v)$. Then, $vw_2w_1y_2y_1 \equiv C_5$, a contradiction.
Therefore, we have $w_2 < w_1$.

Case 4. $y_2 \in N(v)$.
Vertices y_2 and y_1 belong to different components in $\overline{G}[N(v)]$ (otherwise, y_2 belongs to $\mathcal{U}_N$)) and w_2 belongs to $M(v)$.
• If $w_1 \in N(v)$ then, the component containing vertices y_1 and w_1 is impure, and then y_1 belongs to V_1, a contradiction.
• If $w_1 \in M(v)$ then, the component containing y_2 is pseudo-pure, a contradiction.
Therefore, y_2 belongs to $M(v)$.

Case 5 $y_1 \in \mathcal{U}_N$, $y_2 \in \mathcal{R} \cap M(v)$, $w_2 < w_1$.
We again distinguish several cases:

Case 5.1. $w_1, w_2 \in M(v)$.
Vertices y_2, w_1 and w_2 belong to the same component in $G[M(v)]$. Since $y_1 \in \mathcal{U}_N$, there exist a vertex $y \in \mathcal{Y}' \cup \mathcal{Y}''$ and, in the component containing y_1 in $\overline{G}[N(v)]$, an oriented path from y_1 to y. Since $yy_2w_1w_2$ is not an obstruction, we have $y_2 < y$.
• If $y \in \mathcal{Y}'$, then there exist a vertex $x' \in \mathcal{U}_M \setminus N(y)$ and an oriented path $P[y, x']$ such that $yx' \notin E$. We note that $x'y_2 \notin E$ (otherwise, by Claim 10, $x' < y_2$ and $x'y_2P[y, x']$ is a circuit). We consider $(A = abcd, s, t)$, a triplet which is linked with x' ; (A, s, t) is also

linked with vertices y and y_2. If $ay_2 \in E$, then $y_2 \in \mathcal{U}_M$, which contradicts $y_2 \in R'$. So, $ay_2 \notin E$ and y_2yax' is an obstruction in $(G-v,<)$, a contradiction.
• If $y \in \mathcal{Y}''$, then there exist $x \in \mathcal{U}_M$, $x_1 \in N(v)$ and $x_2 \in M(v)$ such that $y < x$, $x_2 < x_1$, $xyx_1x_2 \equiv P_4$, and vertices y, x_1 belong to different components in $\overline{G}[N(v)]$. Note that $y_2x \notin E$ (otherwise, by Claim 10, $x < y_2$ and then yxy_2y is a circuit). We consider $(A = abcd, s, t)$ which is linked with x. (A, s, t) is also linked with vertices y and y_2. By definition of $\mathcal{U}_M$ and $\mathcal{R}$, we have $ay_2 \notin E$. Vertices x_1 and a belong to different components (otherwise, $xx_1 \in E$), thus $ax_1 \in E$. Since xax_1x_2 is not an obstruction, we have $ax_2 \in E$. Vertices y_2 and x_2 are not adjacent (otherwise, since xax_2y_2 is not an obstruction, we have $x_2 < y_2$ and y belongs to a pseudo-pure component). Since $y_2yx_1x_2$ is not an obstruction, we have $x_1y_2 \in E$ and $x_1 < y_2$ (because y_2x_1ax is not an obstruction). Since $yxay$, $x_1y_2yx_1$ and x_1yax_1 are not circuits, we successively have $y < a$, $x_1 < y$ and $x_1 < a$. Since $x_2x_1yax_2$ is not a circuit, we have $x_2 < a$, and then x_2ayy_2 is an obstruction, a contradiction.

Case 5.2. $w_1 \in M(v)$ and $w_2 \in N(v)$.
In this case, v belongs to an induced C_5, a contradiction.

Case 5.3. $w_1 \in N(v)$ and $w_2 \in M(v)$.
Vertices y_1 and w_1 belong to the same component in $\overline{G}[N(v)]$, which is impure (because $w_2 < w_1$). Then, y_1 belong to V_1, a contradiction ($y_1 \notin \mathcal{U}_N$).

Case 5.4. $w_1, w_2 \in N(v)$.
Since w_1 and w_2 are nonadjacent to y_1, vertices w_1, w_2 and y_1 belong to the same component in $\overline{G}[N(v)]$, denoted C. Let us denote Q the component, in $G[M(v)]$, containing y_2. Since $y_1 \in \mathcal{U}_N$, there exist a vertex $y \in \mathcal{Y}' \cup \mathcal{Y}''$ and, in the component C, an oriented path $P[y_1, y]$ from y_1 to y.
• If $y \in \mathcal{Y}'$ then there exist a vertex $x' \in \mathcal{U}_M \setminus N(y)$ and an oriented path $P[y, x']$. Vertices x' and y_2 are nonadjacent (otherwise, by Claim 10, $x' < y_2$ and $x'y_2P[y_1, y]P[y, x']$ is a circuit).
Let $(A = abcd, s, t)$ be linked with x'. (A, s, t) is also linked with vertices y, y_1 and y_2.
If ay_2 is an edge in E, then y_2 belongs to $\mathcal{U}_M$, which contradicts $y_2 \in R'$. Thus, we have $ay_2 \notin E$. Since y_2y_1ax' is not an obstruction in $(G-v,<)$, we have $y_1x' \in E$. Since $y \in \mathcal{U}_N$, y has a neighbour in Q_0. So, by Proposition 1, vertices y and y_1 have the same neighbours in Q_0, which contradicts $x' \in N(y_1) \setminus N(y)$.
• If $y \in \mathcal{Y}''$ then, there exist $x \in \mathcal{U}_M$, $x_1 \in N(v)$ and $x_2 \in M(v)$ such that $y < x$, $x_2 < x_1$, $xyx_1x_2 \equiv P_4$, and vertices x_1, y belong to different components. We have $y_2x \notin E$ (otherwise, by Claim 10, $x < y_2$ and $yxy_2P[y_1, y]$ is a circuit).
Let $(A = abcd, s, t)$ be linked with x. (A, s, t) is also linked with vertices y, y_1 and y_2 (then, $y_1 < a$ and $y < a$). By definitions of $\mathcal{U}_M$ and $\mathcal{R}$, we have $ay_2 \notin E$. Since xay_1y_2 is not an obstruction, we have $y_1x \in E$; then, by Proposition 1, vertices y_1 and y have the same neighbours in Q_0. According to Claim 8, we have ty, $ty_1 \in E$ and $t < y$, $t < y_1$. Since $y_1 \in \mathcal{C}(V_1)$, by Lemma 6, vertices w_2 and y_1 have the same neighbours in Q_0 (then, w_2x, $w_2t \in E$). Since $y_2y_1 \in E$ and $y_2w_2 \notin E$, y_2 does not belong to Q_0 and $Q \neq Q_0$.

Vertices x_1 and a belong to different components in $\overline{G}[N(v)]$ (otherwise, $xx_1 \in E$), then ax_1 is an edge in E. Since xax_1x_2 is not an obstruction, we have $ax_2 \in E$.
If vertices y_2 and x_2 are adjacent, then $x_2 \in Q$ ($Q \neq Q_0$) and $a < x_2$ (because $txax_2$ is not an obstruction). Since xax_2y_2 is not an obstruction, we have $x_2 < y_2$, and then $x_2y_2y_1ax_2$ is a circuit, a contradiction. Thus, y_2 is not adjacent to x_2.
If vertices y_2 and x_1 are adjacent, then $x_1 < y_2$ (because xax_1y_2 is not an obstruction). Since $y_2y_1x_1y_2$, $y_1ax_1y_1$ and ax_2x_1a are not circuits, we successively have $x_1 < y_1$, $x_1 < a$ and $x_2 < a$. Since $txax_2$ is not an obstruction, we have $tx_2 \in E$ and $x_2 \in Q_0$. Since y and y_1 have the same neighbourhood in Q_0 and $x_2y \notin E$, x_2 is nonadjacent to y_1. Then, $x_2ay_1y_2$ is an obstruction, a contradiction. Thus, y_2 is not adjacent to x_1.
Vertices x_2 and y_1 are adjacent (otherwise, $y_2y_1x_1x_2$ is an obstruction) and x_2 does not belong to Q_0 (otherwise, $x_2y \in E$). Since $txax_2$ is not an obstruction, we have $a < x_2$, and since ax_2y_1a is not a circuit, we have $y_1 < x_2$. Therefore, x_1t is an edge in G (because tyx_1x_2 is not an obstruction).
We obtained above, $w_2x, w_2t \in E$. Since $y_2y_1xw_2$ and $y_2y_1tw_2$ are not obstructions, we have $x < w_2$ and $t < w_2$. Moreover, since $xw_2x_1x_2$ is not an obstruction, we have $w_2x_2 \in E$. Then, vertices w_2 and y_1 are distinct from y. Since $y_2y_1aw_2$ and $y_2y_1x_2w_2$ are not obstructions, we have $a < w_2$ and $x_2 < w_2$.
By definition of components C_i in $\overline{G}[N(v)]$, there exists in $\overline{G}[C]$ a chordless path $P = [u_0 = y, ..., u_l]$ which end points are y and $u_l \in \{y_1, w_2\}$. Since $yx_2 \notin E$ and ($u_lx_2 \in E$ and $N_{Q_0}(u_l) \neq \emptyset$), there exists a greater number h ($0 \leq h < l$) such that, either $u_hx_2 \notin E$ or $N_{Q_0}(u_h) = \emptyset$.
We distinguish two cases:

Case 5.4.1. $N_{Q_0}(u_h) = \emptyset$.
Since $t < y_1$ and $t < w_2$, by Lemma 6, we have $u_hy_1, u_hw_2 \in E$ (then, $h < l-1$). We have $a < u_h$ (otherwise, $txau_h$ is an obstruction) and $y_1 < u_h$ (otherwise, $y_1au_hy_1$ is a circuit). Vertices y_2 and u_h are not adjacent, otherwise, either $y_2 < u_h$ and $y_2u_hw_2t$ is an obstruction, or $u_h < y_2$ and $u_hy_2y_1u_h$ is a circuit. Since $w_2u_hy_1y_2$ is not an obstruction, we have $u_h < w_2$.
We consider now the vertex u_{h+1} ($h+1 < l$ and $u_hu_{h+1} \notin E$). By definition of h, we have $u_{h+1}x$ and $u_{h+1}t \in E$. By Lemma 6, we have $u_{h+1} < t$ and $u_{h+1} < x$. Then, we have $u_{h+1}w_2 \in E$ (otherwise, $u_{h+1}xw_2u_h$ is an obstruction), $u_{h+1} < w_2$ (otherwise, $u_{h+1}xw_2u_{h+1}$ is a circuit) and $u_{h+1}y_1 \in E$ (otherwise, $y_1u_hw_2u_{h+1}$ is an obstruction). Since u_{h+1} is adjacent to vertices w_2 and y_1, we have $h+1 < l-1$.
We consider now the vertex u_{h+2} ($h+2 < l$ and $u_{h+1}u_{h+2} \notin E$). By definition of h, we have $u_{h+2}x$ and $u_{h+2}t \in E$. We have $u_{h+2} < u_h$ (otherwise, $u_hu_{h+2}xu_{h+1}$ is an obstruction), $u_{h+2}w_2 \in E$ (otherwise, $u_{h+2}u_hw_2u_{h+1}$ is an obstruction) and $u_{h+2} < w_2$ (otherwise, $w_2u_{h+2}u_hw_2$ is a circuit). Since $u_{h+2}w_2u_{h+1}y_1$ and $u_{h+1}y_1u_hu_{h+2}$ are not obstructions, we have $u_{h+2}y_1 \in E$. Sinceu_{h+2} is adjacent to vertices w_2 and y_1, we have $h+2 < l-1$.
We consider the vertex u_{h+3} ($h+3 < l$ and $u_{h+2}u_{h+3} \notin E$). We have $u_{h+3} < u_{h+1}$ (otherwise, $u_{h+2}u_hu_{h+3}u_{h+1}$ is an obstruction), $u_{h+3}w_2 \in E$ (otherwise, $u_{h+3}u_{h+1}w_2u_{h+2}$ is an obstruction) and $u_{h+3} < w_2$ (otherwise, $w_2u_{h+3}u_{h+1}w_2$ is a circuit). Since $u_{h+3}w_2u_{h+2}y_1$ and $u_{h+2}y_1u_{h+1}u_{h+3}$ are not obstructions, we have $u_{h+3}y_1 \in E$. Since u_{h+3} is adjacent to

vertices w_2 and y_1, we have $h+3<l-1$.
By induction, we obtain that every vertex u_k $(h \leq k < l)$ is adjacent to vertices w_2 and y_1, a contradiction.

Case 5.4.2. $u_hx_2 \notin E$ and $N_{Q_0}(u_h) \neq \emptyset$.
Vertices u_h, y_1 and w_2 have the same neibours in Q_0. We have $u_h < x$ (otherwise, $x_2x_1u_hx$ is an obstruction), $u_hw_2 \in E$ (otherwise, $u_hxw_2x_2$ is an obstruction) and $u_h < w_2$ (otherwise, $u_hxw_2u_h$ is a circuit). Since $u_hw_2x_2y_1$ and $x_2y_1xu_h$ are not obstructions, we have $u_hy_1 \in E$. Since $u_hy_1, u_hw_2 \in E$, we have $h < l-1$.
We consider the vertex u_{h+1} $(h+1 < l$ and $u_hu_{h+1} \notin E)$. By definition of h, we have $u_{h+1}x$, $u_{h+1}t$ and $u_{h+1}x_2 \in E$. We have $u_{h+1} < x_2$ (otherwise, $u_hxu_{h+1}x_2$ is an obstruction), $u_{h+1}w_2 \in E$ (otherwise, $u_{h+1}x_2w_2u_h$ is an obstruction) and $u_{h+1} < w_2$ (otherwise, $w_2u_{h+1}x_2w_2$ is a circuit). Since $u_{h+1}w_2u_hy_1$ and $u_hy_1x_2u_{h+1}$ are not obstructions, we have $u_{h+1}y_1 \in E$. Since $u_{h+1}y_1, u_{h+1}w_2 \in E$, we have $h+1 < l-1$.
We consider the vertex u_{h+2} $(h+2 < l$ and $u_{h+1}u_{h+2} \notin E)$. By definition of h, we have $u_{h+2}x$, $u_{h+2}t$ and $u_{h+2}x_2 \in E$. We have $u_{h+2} < u_h$ (otherwise, $u_{h+1}x_2u_{h+2}u_h$ is an obstruction), $u_{h+2}w_2 \in E$ (otherwise, $u_{h+2}u_hw_2u_{h+1}$ is an obstruction) and $u_{h+2} < w_2$ (otherwise, $w_2u_{h+2}u_hw_2$ is a circuit). Since $u_{h+2}w_2u_{h+1}y_1$ and $u_{h+1}y_1u_hu_{h+2}$ are not obstructions, we have $u_{h+2}y_1 \in E$. Since u_{h+2} is adjacent to vertices w_2 and y_1, we have $h+2 < l-1$.
By induction, we obtain that every vertex u_k $(h \leq k < l)$ is adjacent to vertices w_2 and y_1, a contradiction. $\square$

7 Conclusion

Chvátal [3] defined a graph G to be *brittle* if each induced subgraph F of G contains a vertex that is not a midpoint of any P_4 (semi-simplicial vertex) or not an endpoint of any P_4. If v is not a midpoint of any P_4 or not an endpoint of any P_4 and if $F-v$ is perfectly orderable then F is perfectly orderable. Thus, every brittle graph is perfectly orderable [3].
Let $\{\mathcal{G}_i\}_{\ 1\leq i\leq k}$ be a finite set of finite families of rooted graphs such that for any $i \in \{1,\cdots,k\}$ Assertion A is true. For example, $\mathcal{G}_1 = \{(P_4,v)\}$ where v denotes a midpoint of P_4, $\mathcal{G}_2 = \{(P_4,v)\}$ where v denotes an endpoint of P_4, $\mathcal{G}_3$ is the family depicted in Figure 1 ($\mathcal{G}_3$-free vertices are charming vertices), $\mathcal{G}_4$ is the family in Figure 2 ($\mathcal{G}_4$-free vertices are nice vertices), $\mathcal{G}_5 = \mathcal{F}_2$, $\mathcal{G}_6 = \mathcal{F}_3$ and $\mathcal{G}_7 = \mathcal{F}_4$. We define a class $\mathcal{P}(\mathcal{G}_1,\cdots,\mathcal{G}_k)$ of perfectly orderable graphs in the following way: a graph G belongs to $\mathcal{P}(\mathcal{G}_1,\cdots,\mathcal{G}_k)$ if for any subgraph F of G there exist a family $\mathcal{G}_i$ and a $\mathcal{G}_i$-free vertex v in F. According to the previous example, we note that $\mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)$ is the class of brittle graphs and that $\mathcal{P}(\mathcal{G}_1,\cdots,\mathcal{G}_6) = \mathcal{P}(\mathcal{G}_3,\mathcal{G}_7)$. The last class contains brittle graphs, charming graphs, nice graphs and quasi-brittle graphs.

Although recognizing general perfectly orderable graphs is NP-complete [10], clearly, there is a polynomial time recognition algorithm for $\mathcal{P}(\mathcal{G}_1,\cdots,\mathcal{G}_k)$.

References

[1] C. Berge and P. Duchet, "Strongly Perfect Graphs", in: C. Berge and V. Chvátal, eds., Topics on Perfect Graphs, Annals of Discrete Math. 21 (North Holland, Amsterdam, 1984) 57-61.

[2] V. Chvátal, "Perfectly ordered graphs", in: C. Berge and V. Chvátal, eds., Topics on Perfect Graphs, Annals of Discrete Math. 21 (North Holland, Amsterdam, 1984) 63-65.

[3] V. Chvátal, Perfect Graph Seminar, McGill University, Montreal, Canada (1983).

[4] S. Földes, P. L. Hammer, "Split graphs", Proc. 8th S-E Conf. Combinatorics, Graphs and Computing, Congressus Numerantium XVII (1977) 311-315.

[5] J.L. Fouquet, F. Roussel, P. Rubio, H. Thuillier, "New classes of perfectly orderable graphs.", to appear in Discr. Math.

[6] M. C. Golumbic, "Algorithmic Graph Theory and Perfect Graphs", Academic Press, New York, 1980.

[7] B. Jamison and S. Olariu, "On the Semi-perfect Elimination", Advances in Applied Mathematics 9 (1988) 364-376.

[8] C. T. Hoàng and N. Khouzam, "On Brittle Graphs", J. Graph Theory 12 (3) (1988) 391-404.

[9] C. T. Hoàng, F. Maffray, S. Olariu and M. Preissmann, "A charming class of perfectly orderable graphs", Discrete Mathematics 102 (1992) 67-74.

[10] M. Middendorf and F. Pfeiffer, "On the complexity of recognizing perfectly orderable graphs", Discrete Mathematics 80 (1990) 327-333.

[11] S. Olariu, "Quasi-brittle graphs, a new class of perfectly orderable graphs", Discrete Mathematics 113 (1992) 143-153.

[12] P. L. Hammer, B. Simeone, "The splittance of a graph", Combinatorica 1 (3) (1981) 275-284.

[13] C. T. Hoàng, B. A. Reed, "Some Classes of Perfectly Orderable Graphs", J. Graph Theory 13 (4) (1989) 445-463.

[14] C. T. Hoàng, Ph.D. Thesis, McGill University, Montreal, Canada (1985).