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Partitionable graphs arising from near-factorizations of finite groups

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Abstract

In 1979, two constructions for making partitionable graphs were introduced in [7].

The graphs produced by the second construction are called CGPW graphs. A near-factorization (A, B) of a finite group is roughly speaking a non-trivial factorization of G minus one element into two subsets A and B. Every CGPW graph with n vertices turns out to be a Cayley graph of the cyclic group $\mathbf{Z}_n$, with connection set $(A - A) \setminus \{0\}$, for a near-factorization (A, B) of $\mathbf{Z}_n$. Since a counter-example to the Strong Perfect Graph Conjecture would be a partitionable graph [12], any 'new' construction for making partitionable graphs is of interest.

In this paper, we investigate the near-factorizations of finite groups in general, and their associated Cayley graphs which are all partitionable. In particular we show that near-factorizations of the dihedral groups produce every CGPW graph of even order. We present some results about near-factorizations in finite groups which imply that a finite abelian group with a near-factorization (A, B) such that $|A| \leq 4$ must be cyclic (already proved in [5]). One of these results may be used to speed up exhaustive calculations. At last, we prove that there is no counter-example to the Strong Perfect Graph Conjecture arising from near-factorizations of a finite abelian group of even order.

Keywords: partitionable, perfect, graph, near-factorization, group

1 Introduction

In 1960, Claude Berge introduced the notion of perfect graphs : a graph is *perfect* if for every induced subgraph H of it, the chromatic number of H does not exceed the maximum number of pairwise adjacent vertices in H. A *hole* is a chordless cycle with at least four vertices. Berge conjectured that perfect graphs are exactly the graphs with no induced odd holes and no induced complement of an odd hole, or equivalently that *minimal imperfect graphs* are odd holes and their complements. This conjecture is often called the Strong Perfect Graph Conjecture and has motivated many works.

Following the paper of Bland, Huang and Trotter [3], a graph G is said to be *partitionable* if there exist two integers p and q such that G has $pq + 1$ vertices and for every vertex v of G, the induced subgraph $G \setminus \{v\}$ admits a partition in p cliques of cardinality q and also admits a partition in q stable sets of cardinality p. Let ω denote the maximum cardinality of a clique of G and α denote the maximum cardinality of a stable set of G. Then it is known that $p = \alpha$ and $q = \omega$.

Padberg [12] proved that every minimal imperfect graph is partitionable. Thus a counter-example to the Strong Perfect Graph Conjecture would lie in the class of partitionable graphs. Hence an approach to Berge's conjecture is to prove that a given class of partitionable graphs does not contain any minimal imperfect graph which is not an odd odd hole or anti-hole.

In 1979, Chvátal, Graham, Perold and Whitesides introduced two constructions for making partitionable graphs [7]. In 1996, Sebö proved that there is no counter-example to the Strong Perfect Graph Conjecture in the first one [13]. In 1984, Grinstead proved that there is no counter-example to the Strong Perfect Graph Conjecture in the second one [10]. A *variant* of a partitionable graph is a partitionable graph with the same vertices, the same maximum cliques and the same maximum stable sets. In 1998, Bacsó, Boros, Gurvich, Maffray and Preissmann [1] extended Grinstead's result to the wider class of the variants of the second construction.

A graph with n vertices is *circular* if there exists a cyclic numbering of its vertices (modulo n) such that, for every vertex x, for every maximum clique C and for every maximun stable set S, the set $\{(c+x) \pmod{n} \mid c \in C\}$ is a maximum clique and the set $\{(s + x) \pmod{n} \mid s \in S\}$ is a maximun stable set.

A *normalized* graph is a graph such that for every edge $\{i, j\}$, there exists a maximum clique containing both i and j.

A partitionable graph produced by the second construction due to Chvátal, Graham, Perold and Whitesides is called a *CGPW* graph, where CGPW graph is the abbreviation of Chvátal-Graham-Perold-Whithesides graph. Any CGPW graph appears to be a circular normalized partitionable graph. Grinstead conjectured in 1984 that the converse is true, that is, any circular normalized partitionable graph is a CGPW graph. This question is still open, though it has been proved for graphs with size of maximum cliques less than 5 by Bacsó, Boros, Gurvich, Maffray and Preissmann [1].

Let G be a finite group of order n with operation $*$. Two subsets A and B of G of cardinality at least 2 are said to form a *near-factorization* of G if and only if $n = |A| \times |B| + 1$ and there is an element $u(A, B)$ of G such that $A * B = G \setminus \{u(A, B)\}$. Let S be a symmetric subset of G which does not contain the identity element e. The *Cayley graph* with *connection set* S is the graph with vertex set G and edge set $\{\{i, j\},\ i^{-1} * j \in S\}$. We denote by $\mathrm{Cay}(G, S)$ this graph. Notice that the definitions of a Cayley graph given in the litterature may differ. The one we use in this paper is very close from the definition given in the book 'Algebraic Graph Theory' of Norman Biggs [2]. Since S is a symmetric set such that $e \notin S$, the graph $\mathrm{Cay}(G, S)$ is a simple graph without loops, as are all graphs in this paper.

Let Γ be any circular normalized partitionable graph with n vertices. Let C be a maximum clique of Γ and let S be a maximum stable set of Γ. Then it is easy to see that (C, S) is a near-factorization of the group Z_n and that Γ is the Cayley graph of the finite group Z_n with connection set $(C - C) \setminus \{0\}$. The converse is true: if (A, B) is a near-factorization of Z_n then the Cayley graph with connection set $(A - A) \setminus \{0\}$ is a circular normalized partitionable graph [1].

Due to this equivalence, the second construction of Chvátal, Graham, Perold and Whitesides had been first described by N.G. De Bruijn in 1956 [8].

If (A, B) is a near-factorization of a finite group then the Cayley graph with connection set $(A^{-1} * A) \setminus \{e\}$ is a normalized partitionable graph (Section 2). This observation has motivated this paper: the main aim is to produce near-factorizations of some finite groups, so as giving rise to 'new' partitionable graphs. We give 'new' near-factorizations for the dihedral groups but the associated Cayley graphs turn out all to be CGPW graphs (Section 3). These near-factorizations produce all CGPW graphs of even order. In Section 2, we give several results about near-factorizations for finite groups in general, which may be used to speed up exhaustive searches by computer. We give tools to explain why many groups do not have any near-factorization at all. We also prove that no Cayley graph associated to a near-factorization of an abelian group of even order is a counter-example to the Strong Perfect Graph Conjecture.

2 Near-factorizations of finite groups and partitionable graphs.

A *group* is a non-empty set G with a closed associative binary operation $*$, an identity element e, and an inverse a^{-1} for every element $a \in G$. If G has a finite number of elements, then the cardinality of G is denoted by $|G|$ and is called the *order* of G. To avoid a conflict of notation, we use the symbol $\times$ to denote the standard multiplication between two integers. An *abelian* group is a group G such that $*$ is commutative, that is $g * g' = g' * g$ for all elements g and g' of G.

If X and Y are two subsets of G, we denote by $X * Y$ the set $\{x * y,\ x \in X,\ y \in Y\}$. With a slight abuse of notation, if g is an element of G and X is subset of G, we denote by gX the set $\{g\} * X$ and Xg the set $X * \{g\}$. Furthermore $|X|$ is the cardinal of X, that is the number of elements of X. The subset X is said to be *symmetric* if $X = X^{-1}$, where X^{-1} is the set $\{x^{-1},\ x \in X\}$.

Recall that two subsets A and B of cardinal at least 2 of a finite group G of order n form a *near-factorization* of G if and only if $n = |A| \times |B| + 1$ and there is an element $u(A, B)$ of G such that $A * B = G \setminus \{u(A, B)\}$: $u(A, B)$ is called the *uncovered element* of the near-factorization. Sometimes, we shall write simply u instead of $u(A, B)$. The condition about the cardinal of A and B is required to avoid the trivial case $A = G \setminus \{u\}$ and $B = \{e\}$. Notice that every element x of G distinct from u may be written in a unique way as $x = a * b$ with $a \in A$ and $b \in B$. Hence a near-factorization (A, B) may be seen as a tiling of $G \setminus \{u(A, B)\}$ with proto tile A.

The *cyclic group* of order n is the group which is generated by an element x of order n. This group is denoted by $\mathbf{Z}_n$. For conveniency, we use the following representation of $\mathbf{Z}_n$: the elements of $\mathbf{Z}_n$ are the integers between 0 and $n - 1$ and the operation $*$ is defined by $x * y = (x + y) \pmod{n}$. Due to this definition of the operation of $\mathbf{Z}_n$, we denote this operation by $+$ rather than $*$.

Example 1

Let $\mathbf{Z}_{13}$ be the cyclic group of order 13,

Let $A = \{0, 1, 2\}$ and $B = \{0, 3, 6, 9\}$.

Then $A + 0 = \{0, 1, 2\}$, $A + 3 = \{3, 4, 5\}$, $A + 6 = \{6, 7, 8\}$ and $A + 9 = \{9, 10, 11\}$. Thus $A + B = (Z_{13} \setminus \{12\})$, that is (A, B) is a near-factorization of Z_{13}.

The following figure shows the tiling of $Z_{13} \setminus \{12\}$ given by (A, B).

$A = \{0, 1, 2\} \quad u = 12$

$B = \{0, 3, 6, 9\}$

$n = 13$

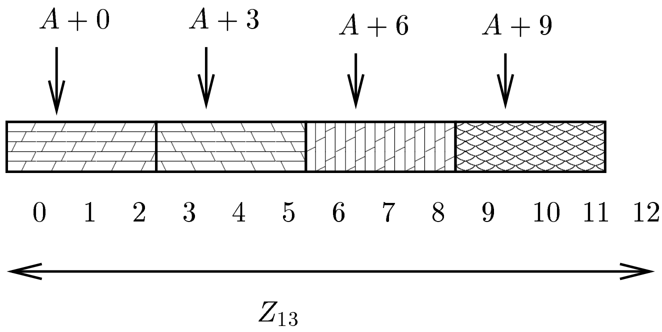


Figure 1: Example of a near-factorization of $\mathbf{Z}_{13}$

Note that if A and B are seen as sets of integers and $+$ denotes the usual addition between integers, then $A + B$ is a tiling of the segment $[0, 11]$. This connection is somewhat detailed in section 3.

The *dihedral group D_{2n}* of even order $2 * n$ (with $n \geq 3$) is the non-abelian group generated by two elements r and s such that:

- r is of order n.
- s is of order 2.
- $s * r = r^{-1} * s$

The problem of characterizing the near-factorizations of the dihedral groups is addressed in Section 3.

Let $g_1, \cdots, g_n$ be the elements of the group G with $g_1 = e$. If R is any subset of G, we denote by $M(R)$ the square $n \times n$ $(0,1)-$matrix defined by $M(R)_{i,j} = 1$ if and only if $g_j \in g_i R$.

Let I be the $n \times n$ identity matrix and J be the $n \times n$ matrix with all entries equal to 1. Then De Caen, Gregory, Hughes and Kreher [5] observed that (A, B) is a near-factorization of G with uncovered element e if and only if $M(A)M(B) = J - I$.

Since $M(A)M(B) = J - I$ implies that $M(B)M(A) = J - I$ (Theorem 1 [5], or [7]). We have the following property:

Lemma 1 *[5] Let G be a finite group and A, B be two subsets of G. Then (A, B) is a near-factorization of G with $u(A, B) = e$ if and only if (B, A) is a near-factorization of G with $u(B, A) = e$.*

The hypothesis $u(A, B) = e$ is actually necessary: consider the dihedral group D_{16} of order 16. Let $A = \{e, r^5, sr^5\}$ and $B = \{e, s, r, sr, sr^7\}$. A small calculation shows that $A * B = D_{16} \setminus \{r^7\}$. Thus (A, B) is a near-factorization of D_{16}, though (B, A) is not one as $sr^5 = e * sr^5 = s * r^5$.

The graph *G(A,B)* associated with a near-factorization (A, B) is the Cayley graph with connection set $(A^{-1} * A) \setminus \{e\}$.

If Γ is a graph, we denote by $\omega(\Gamma)$ the maximum cardinal of a clique of Γ and $\alpha(\Gamma)$ the maximum cardinal of a stable set of Γ. We denote by $V(\Gamma)$ the vertex set of Γ and $E(\Gamma)$ the edge set of Γ.

The graph Γ with vertex set V is *isomorphic* to the graph Γ' with vertex set V' if there exists a bijective map f from V onto V' such that $\{i, j\}$ is an edge of Γ if and only if $\{f(i), f(j)\}$ is an edge of Γ'.

If e' is an edge of Γ we denote by $\Gamma - e'$ the subgraph of Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \setminus \{e'\}$. Likewise, if e' is a non-edge of Γ, we denote by $\Gamma + e'$ the graph with vertex set $V(\Gamma)$ and with edge set $E(\Gamma) \cup \{e'\}$. If v is any vertex of Γ, we denote by $\Gamma \setminus \{v\}$ the induced subgraph of Γ with vertex set $V(\Gamma) \setminus \{v\}$ and edge set $\{\{x, y\} \mid \{x, y\} \in V(\Gamma),\ x \neq v,\ y \neq v\}$.

Obviously, distinct near-factorizations of a given group may give rise to the same graph. In particular, we may left-shift A and right-shift B without altering the associated graph:

Lemma 2 *Let x and y be two elements of G. Then (xA, By) is a near-factorization of G such that $u(xA, By) = x * u(A, B) * y$ and $G(xA, By)$ is isomorphic to $G(A, B)$.*

Proof. The proof is straightforward. □

We say that (xA, By) is *shift-isomorphic* to (A, B).

Thus due to Lemma 2, we may always assume that the uncovered element is e, without altering the associated graph.

In the case of abelian groups, De Caen, Gregory, Hughes and Kreher gave a useful property of near-factorizations:

Lemma 3 *[5] Let G be an abelian group and (A, B) be a near-factorization of G. Then there exist two elements x and y of G such that xA is symmetric and that By is symmetric.*

An *automorphism* of G is a bijective map h of G onto itself such that $h(x * y) = h(x) * h(y)$ for all x and y of G. An *inner-automorphism* h of G is an automorphism of G such that there exists an element g of G which satisfies $h(x) = g * x * g^{-1}$ for all x of G.

Then we have this obvious Lemma:

Lemma 4 *Let* Cay(G,S) *be a Cayley graph with connection set S of a group G. Let h be any automorphism of G. Then the Cayley graph* Cay(G,h(S)) *is isomorphic to* Cay(G,S).

If y is any element of G, we denote by $\langle y \rangle$ the cyclic subgroup of G generated by y. The *order* of y is the smallest integer k such that $y^k = e$ and is denoted by $o(y)$. An *involution* of G is an element of G of order 2. The *center* of G is the set of all elements in G which commute with every element of G.

Let H be any subgroup of G and (A, B) be a near-factorization of G with uncovered element u.

A *right coset* of H is any subset Hx with $x \in G$. A *left coset* of H is any subset xH with $x \in G$. Lagrange's Theorem asserts that for any subgroup H of G, there exists a unique partition of G in right cosets of H. Likewise there exists a unique partition in left cosets of H. A subgroup H of G is *normal* if for every g of G, we have $gH = Hg$.

A *right-tile* of A is the trace of A onto a right-coset of H, that is the subset T is a right-tile of A if and only if there exists g in G such that $T = A \cap Hg$. A *left-tile* of A is the trace of A onto a left-coset of $H\cdot$

The unique partition of G in right cosets of H induces a unique partition of A in right-tiles: let $\{Hg_1, \ldots, Hg_d\}$ be the partition of G in right-cosets, then the set of right-tiles of A is $\{A \cap Hg_1, \ldots, A \cap Hg_d\}$. If T is a right-tile of A which is equal to a whole right-coset, then T is called a *H-right-coset.*

Let τ be the partition of A in right-tiles induced by a given subgroup H. Clearly $\{Tb,\ T \in \tau,\ b \in B\}$ is a partition of $G \setminus \{u\}$. Hence, given the subgroup H, a near-factorization (A, B) may be seen as a tiling of $G \setminus \{u\}$ with the right-tiles of A as tiles. Let K be any such tile and b be any element of B. Notice that Kb lies entirely in a right-coset of H. Thus this tiling of $G \setminus \{u\}$ induces a tiling for every right-coset of H distinct from Hu and induces a tiling of $(Hu) \setminus \{u\}$. Let Hg be any right coset of H: we shall say that the right-tile K is used to cover Hg if there exists an element b of B such that $Kb \subseteq Hg$. The trick of many proofs in this paper is to collect enough informations about the tiling of every right-coset of H so as being able to get informations about the near-factorization (A, B).

Example 2 ———

Let (A, B) be the near-factorization of the dihedral group D_{16} given by $A = \{e, r^5, sr^5\}$ and $B = \{s, r, sr, r^2, sr^2\}$.

Let $H := \{e, s\}$ be the cyclic subgroup of D_{16} generated by s. Then $\{H, Hr, Hr^2, \cdots, Hr^7\}$ is the partition of D_{16} in right cosets of H. Hence A splits in exactly two right-tiles T_1 and T_2 with

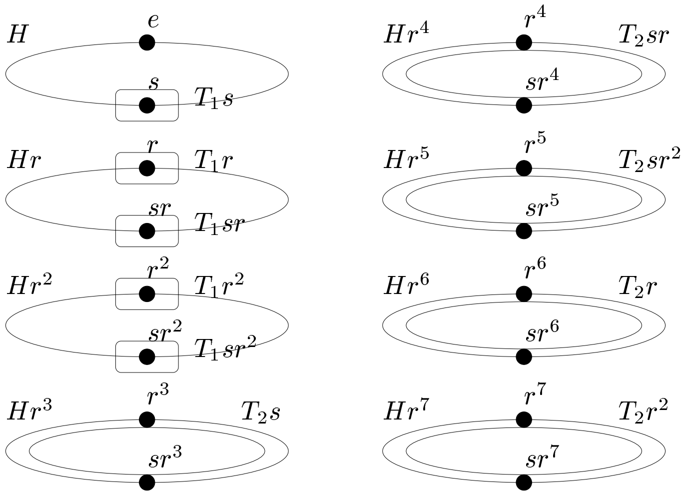
$$T_1 = \{e\} = A \cap H$$

$$T_2 = \{r^5, sr^5\} = A \cap Hr^5$$

The tile T_2 is a H-right-coset. The set B has 5 elements, this implies that T_2 is used to cover 5 of the 8 right-cosets of H, namely the right-cosets Hr^3, Hr^6, Hr^4, Hr^7 and Hr^5 because $Hr^3 = T_2s$, $Hr^6 = T_2r$, $Hr^4 = T_2sr$, $Hr^7 = T_2r^2$ and $Hr^5 = T_2sr^2$.

The tile T_1 is used exactly twice to cover the right-coset Hr as $Hr = \{r, sr\} = T_1r \cup T_1sr$. The tile T_1 is used exactly twice to cover the right-coset Hr^2 as $Hr^2 = \{r^2, sr^2\} = T_1r^2 \cup T_1sr^2$. The last time T_1 is used, it is to cover $H \setminus \{e\}$ as $H \setminus \{e\} = \{s\} = T_1s$.

The following figure represents this tiling of the right-cosets of H.



The unique partition of G in left cosets of H also induces a unique partition of A in left-tiles. If T is a left-tile of A which is equal to a whole left-coset, then T is called a *H-left-coset*.

When the uncovered element is e, we know that (B, A) is a near-factorization of G too. Thus we get a tiling of $G \setminus \{e\}$ with the left-tiles of A as tiles. Let K be any such tile and b be any element of B. Notice that bK lies entirely in a left-coset of H. Hence we have a tiling for every left-coset of H distinct from He and a tiling of $(He) \setminus \{e\}$. Let gH be any left coset of H: we shall say that the left-tile K is used to cover gH if there exists an element b of B such that $bK \subseteq gH$.

Example 3

We consider again the near-factorization $(A, B)$of the dihedral group D_{16} given by $A = \{e, r^5, sr^5\}$ and $B = \{s, r, sr, r^2, sr^2\}$ and the cyclic subgroup H of D_{16} generated by s.

As $u(A, B) = e$, we know that (B, A) is a near-factorization of D_{16} too.

Note that $\{H, rH, r^2H, \cdots, r^7H\}$ is the partition of D_{16} in left cosets of H. Hence A splits in exactly three left-tiles T_1, T_2 and T_3 with

$$T_1 = \{e\} = H \cap A$$

$$T_2 = \{r^5\} = r^5H \cap A$$

$$T_3 = \{sr^5\} = r^3H \cap A$$

Thus no left-tile of A is a left-coset. This means that the tiling induced by (B, A) is actually different of the one induced by (A, B).

Let $Hg_1, Hg_2, \ldots, Hg_d$ be a partition of G in right-cosets of H. Let X be any subset of G. We define the integer $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(X)$ as

$$\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(X) := |\{i,\ 1 \le i \le d,\ \emptyset \subsetneq Hg_i \cap X \subsetneq Hg_i\}|$$

The counter $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(X)$ is the number of right-cosets of H which meet X and are not a subset of X.

Let $\mathrm{disp}^{\mathrm{l}}_{\mathrm{H}}(X)$ be the number of left-cosets of H which meet X and are not a subset of X. When H is a normal subgroup then we use rather the notation $\mathrm{disp}_{\mathrm{H}}(X)$ instead of $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(X)$ or $\mathrm{disp}^{\mathrm{l}}_{\mathrm{H}}(X)$. The notation $\mathrm{disp}_{\mathrm{H}}$ is related to the word 'dispersion'.

Let y be any element of G. A subset W of G is a *left-y-chain* (respectively *right-y-chain*) if $|W| \neq |\langle y \rangle|$ and W can be written $w * \{e,\ y,\ \ldots,\ y^{|W|-1}\}$ (respectively $\{e,\ y,\ \ldots,\ y^{|W|-1}\} * w$).

If H is a cyclic subgroup $\langle y \rangle$, then it is useful to subdivide any tile of A in right-y-chains. For conveniency, these right-y-chains will be considered again as tiles. Let $T := \{e,\ y,\ \ldots,\ y^{|T|-1}\} * t$ and $T' := \{e,\ y,\ \ldots,\ y^{|T'|-1}\} * t'$ be two maximal right-y-chains of A not necessarily distinct. Let b and b' be two elements of B. The tile $T'b'$ is said to be used after the tile Tb if and only if $t' * b' = y^{|T|} * t * b$. This implies that $t'^{-1} * y^{|T|} * t = b' * b^{-1}$ is an element of $B * B^{-1}$. When this relation is all we need, we say simply that the tile T' is used after the tile T (see figure 2).

The fact that $G(A, B)$ is a normalized partitionable graph may be deduced from [7] and [5]. We give here a direct proof which shows how the near-factorization (A, B) and the partitionable graph are closely related, by exhibiting the partition in maximum cliques and the partition in maximum stable sets of $G(A, B) \setminus \{x\}$ for every x:

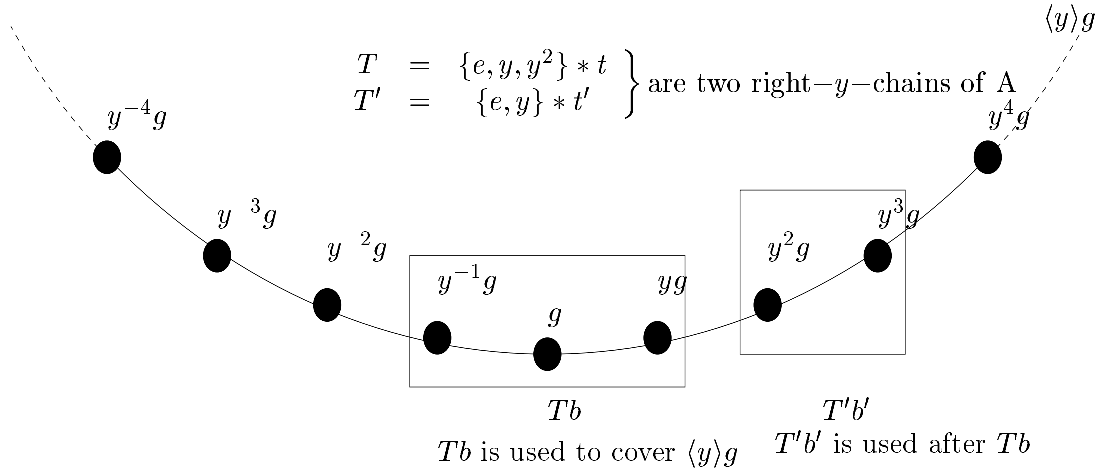


Figure 2: Fragment of the tiling of the coset $\langle y\rangle g$

$$\left.\begin{array}{rcl} tb & = & y^{-1}g \\ t'b' & = & y^2 g \end{array}\right\} \Rightarrow b' = t'^{-1}y^3 tb$$

Lemma 5 *If (A, B) is a near-factorization of a finite group G such that $A * B = G \setminus \{e\}$, then the graph* G(A,B) *is a normalized partitionable graph with maximum cliques $\{xA,\ x \in G\}$ and maximum stable sets $\{xB^{-1},\ x \in G\}$.*

Proof.

Fact 1 *For every x of G, xA is a clique of $G(A, B)$.*

Let x_1 and x_2 be two distinct elements of xA: there exist a_1 and a_2 of A such that $x_1 = x * a_1$ and $x_2 = x * a_2$. Then ${x_1}^{-1} * x_2 = {a_1}^{-1} * a_2$ is an element of $(A^{-1} * A) \setminus \{e\}$. Thus $\{x_1, x_2\}$ is an edge of $G(A, B)$, and so xA is a clique of $G(A, B)$.

Fact 2 *For every x of G, xB^{-1} is a stable set of $G(A, B)$.*

Let x_1 and x_2 be two distinct elements of xB^{-1}: there exist b_1 and b_2 of B such that $x_1 = x * {b_1}^{-1}$ and $x_2 = x * {b_2}^{-1}$.

If $\{x_1, x_2\}$ is an edge of $G(A, B)$, then ${x_1}^{-1} * x_2 = b_1 * b_2^{-1}$ is an element of $A^{-1} * A$. Thus there exist a_1 and a_2 in A such that $b_1 * b_2^{-1} = {a_1}^{-1} * a_2$. Hence $a_1 * b_1 = a_2 * b_2$. Since (A, B) is a near-factorization, this implies that $a_1 = a_2$ and $b_1 = b_2$. Thus $x_1 = x_2$, a contradiction.

Hence $\{x_1, x_2\}$ is not an edge of $G(A, B)$. This implies that xB^{-1} is a stable set of $G(A, B)$.

Fact 3 *For every x of G, $G(A, B) \setminus \{x\}$ is partitionned by the $|B|$ cliques $\{xbA,\ b \in B\}$ and is also partitionned by the $|A|$ stable sets $\{xa^{-1}B^{-1},\ a \in A\}$. Hence $G(A, B)$ is a partitionable graph with $\omega = |A|$ and $\alpha = |B|$.*

If there exists b in B such that $x \in xbA$ then there is an element a in A such that $x = x * b * a$ thus $e = b * a$, which is in contradiction with Lemma 1 and the definition of a near-factorization. Hence $\bigcup_{b \in B} xbA \subset G \setminus \{x\}$. If $xbA \cap xb'A \neq \emptyset$ with b and b' in B, then there are a and a' in A such that $x * b * a = x * b' * a'$ thus $b * a = b' * a'$. This implies with Lemma 1 again that $a = a'$ and $b = b'$. Hence $|\bigcup_{b \in B} xbA| = \sum_{b \in B} |xbA| = |B| * |A| = |G \setminus \{x\}|$. Thus $\bigcup_{b \in B} xbA = G \setminus \{x\}$ and $\{xbA,\ b \in B\}$ is a partition of $G \setminus \{x\}$.

If there exists a in A such that $x \in xa^{-1}B^{-1}$ then there is an element b in B such that $x = x * a^{-1} * b^{-1}$ thus $e = a^{-1} * b^{-1}$ and so $e = b * a$: contradiction. Hence $\bigcup_{a \in A} xa^{-1}B^{-1} \subset G \setminus \{x\}$. If $xa^{-1}B^{-1} \cap xa'^{-1}B^{-1} \neq \emptyset$ with a and a' in A, then there are b and b' in B such that $x * a^{-1} * b^{-1} = x * a'^{-1} * b'^{-1}$ thus $a^{-1} * b^{-1} = a'^{-1} * b'^{-1}$ and so $b * a = b' * a'$. This implies that $a = a'$ and $b = b'$. Hence $|\bigcup_{a \in A} xa^{-1}B^{-1}| = \sum_{a \in A} |xa^{-1}B^{-1}| = |B| * |A| = |G \setminus \{x\}|$. Thus $\bigcup_{a \in A} xa^{-1}B^{-1} = G \setminus \{x\}$ and $\{xa^{-1}B^{-1},\ a \in A\}$ is a partition of $G \setminus \{x\}$.

Fact 4 *For every maximum clique Q of $G(A,B)$, there is an element x of G such that $Q = xA$, hence the set of the n maximum cliques is $\{xA,\ x \in G\}$. Likewise the set of the n maximum stable sets of $G(A,B)$ is $\{xB^{-1},\ x \in G\}$.*

Since $G(A,B)$ is a partitionable graph, we know that $G(A,B)$ has exactly n maximum cliques. Thus we are done if we show that for every pair of elements x and y of G such that $x \neq y$, we have $xA \neq yA$. This is equivalent to show that if $A = zA$ then $z = e$. Suppose $A = zA$. Then for every element a of A, we have that $z * a$ is an element of A. Thus A admits a partition in right-$\langle z\rangle$-cosets. Hence $\omega = 0 \pmod{o(z)}$ where $o(z)$ is the order of z. Thus $n = 1 \pmod{o(z)}$. As $o(z)$ divides the number of elements of G, we also have $n = 0 \pmod{o(z)}$. Therefore $o(z) = 1$ and so $z = e$. This proof also works for the maximum stable sets.

Fact 5 *$G(A,B)$ is a normalized graph.*

Let $\{x,y\}$ be any edge of $G(A,B)$. Then $x^{-1} * y \in A^{-1} * A$, thus there exists $a \in A$ such that $y \in xa^{-1}A$. Obviously $x \in xa^{-1}A$. Hence $G(A,B)$ is a normalized graph.
□

Since the cardinal of a maximum clique of $G(A,B)$ is equal to $|A|$, we denote by ω the value of $|A|$. Likewise, we denote by α the value of $|B|$.

A graph $\Gamma = (V,E)$ on $\alpha\omega + 1$ vertices is called a *web*, if the maximum cardinal of a clique of Γ is ω, the maximum cardinal of a stable set of Γ is α, and there is a cyclical order of V so that every set of ω consecutive vertices in this cyclical order is an $\omega-$clique. Equivalently, normalized webs with n vertices are graphs induced by any near-factorization (A,B) of Z_n such that A is an interval.

Let X be any subset of the group G. We set

$$\mathrm{INT}(X) = \max_{x\in G,\ y\in G,\ x\neq y} \{|xX \cap yX|\}$$

.

Notice that $\mathrm{INT}(A)$ denotes the maximum cardinal of the intersection between two distinct $\omega-$cliquesof $G(A,B)$ and that $\mathrm{INT}(B^{-1})$ denotes the maximum cardinal of the intersection between two distinct $\alpha-$stable sets.

An edge e of a graph Γ is said to be an $\alpha - critical\ edge$ if and only if $\alpha(\Gamma - e) > \alpha(\Gamma)$. Similarly, a non-edge e' is said to be $co - critical$ if and only if $\omega(\Gamma + e') > \omega(\Gamma)$. It is easy to check that a graph $G(A,B)$ has a $co - critical$ non-edge (respectively $\alpha - critical$ edge) if and only if $\mathrm{INT}(A) = \omega - 1$ (respectively $\mathrm{INT}(B^{-1}) = \alpha - 1$).

Lemma 6

$$\mathrm{INT}(X) = \max_{g\in G\setminus\{e\}} \{|X \cap gX|\}$$

Proof. The proof is straightforward. □

Next lemma will be used in the proofs of this article:

Lemma 7 *Let G be a finite group having a near-factorization (A,B). Let H be any normal subgroup of G. If there is a H-coset (Ha) in A, then in every coset of H, a tile T of A may be used at most once.*

Proof. Let T be any tile of A: there exists y of G such that $T = A \cap Hy$. Let g be any element of G and let B_g be the set $\{b \in B,\ Tb \subseteq Hg\}$. We want to show that $|B_g| \leq 1$.

If $|B_g| \geq 2$ then there exist two distinct elements b and b' of B such that $Tb \subseteq Hg$ and $Tb' \subseteq Hg$. From $T \subseteq Hy$, we get $Hg = Hyb$ and $Hg = Hyb'$. Then $Hab = ay^{-1}Hyb$ because H is a normal subgroup. Thus $Hab = ay^{-1}Hg = ay^{-1}Hyb' = Hab'$. Since (A,B) is a near factorization and $Ha \subseteq A$, $\{b,b'\} \subseteq B$, this implies that $b = b'$: a contradiction. Hence $|B_g| \leq 1$. □

Note that example 2 shows that the hypothesis that H must be normal is actually needed.

We are now ready to state the main result of this paper.

Theorem 1 *Let G be a finite group admitting a symmetric near-factorization (A,B). Let H be a non-trivial proper subgroup of G. Then*

1. $\mathrm{disp^r_H}(A) > 0$ *and* $\mathrm{disp^l_H}(A) > 0$.
2. *if* $\mathrm{disp^r_H}(A) = 1$ *or* $\mathrm{disp^l_H}(A) = 1$ *then* $|H| = 2$.
3. *if H is a normal subgroup,* $\mathrm{disp_H}(A) = 2$ *and* $|A| \neq 2$ *then* $|H| = \frac{n}{2}$.

Proof. Since no special property is required for B, we may assume that $u(A,B)=e$ since otherwise all we have to do is to right-shift B by $u(A,B)^{-1}$. Hence we have $A*B=G\setminus\{e\}=B*A$ (Lemma 1).

1. if $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A)=0$, then every right-tile of A is a H-right-coset. Let T be a right-tile of A which is used to cover the right-coset He. There exists b of B such that $Tb\subseteq He$. Since T is a H-right-coset, we have $Tb=He$. Hence $e\in A*B$, a contradiction. Thus $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A)>0$.

   Likewise, we have $\mathrm{disp}^{\mathrm{l}}_{\mathrm{H}}(A)>0$.

2. Suppose that $\mathrm{disp}_{\mathrm{H}}(A)=1$. Let Hg_1, Hg_2, $\ldots$, Hg_d be a partition of G in right-cosets of H. Since $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A)=1$ there exists a unique integer p between 1 and d such that $\emptyset\subsetneq A\cap Hg_p\subsetneq Hg_p$. Let $A':=A\cap Hg_p$. Thus the set of right-tiles of A is A' and some H-right-cosets.

   Let b be an element of B such that $A'b\subseteq He$. Then we have $Hg_pb=He$, which implies that $(g_p*b)\in He$. Thus, if for every b in B, we have $A'b\subseteq He$, then $g_pB\subseteq He$. We know that (B,A) is a near-factorization with $u(B,A)=e$. Hence (g_pB,A) is a near-factorization with uncovered element g_p. As $g_pB\subseteq He$, g_pB has only one right-tile. Since H is a proper subgroup of G, there exists a right coset Hx distinct from He. Thus $|Hx|=0 \pmod{|g_pB|}=0 \pmod{\alpha}$, which implies $n=0 \pmod{\alpha}$, contradicting the relation $n=\alpha\times\omega+1$.

   Hence there exists b in B such that $A'b$ lies in a coset Hx distinct from He. Obviously A' is the only tile of A which can be used to cover Hx because the other tiles are H-right-cosets thus $|Hx|=0 \pmod{|A'|}$. The tile A' is again the only tile which can be used to cover He, thus $|He|=1 \pmod{|A'|}$. Hence $|A'|=1$.

   Let H' be the conjugate subgroup $g_p^{-1}Hg_p$ of H. Let $H'g'_1$, $H'g'_2$, $\ldots$, $H'g'_d$ be a partition of G in right-cosets of H'. For every i between 1 and d, let $B_i:=B\cap H'g'_i$. Then for every i between 1 and d, we have $(A'*B_i)\subseteq(Hg_p*g_p^{-1}Hg_pg'_i)=Hg_pg'_i$.

   Let i be any integer between 1 and d. If $B_i\neq\emptyset$ then A' is used at least once to cover $Hg_pg'_i$. Thus $Hg_pg'_i$ is covered with the right-tile A' only. Hence we have $(Hg_pg'_i)\setminus\{e\}=\cup_{b\in B,\ A'b\subseteq Hg_pg'_i}A'b$. Let b be any element of B and let j be the integer such that $b\in B_j$. Thus $A'b_j\subseteq Gg_pg'_j$. Hence, if b is not in B_i then $A'b$ is not a subset of $Hg_pg'_i$. Thus we have $A'*B_i=(Hg_pg'_i)\setminus\{e\}$. Since $|A'|=1$, we must have $|B_i|=|(Hg_pg'_i)\setminus\{e\}|$.

   Hence we have for all i between 1 and d, $|B_i|=0$ or $|B_i|=|Hg_pg'_i\setminus\{e\}|$. Thus $\mathrm{disp}_{\mathrm{d}\,H'}(B)\leq 1$. We know that $\mathrm{disp}_{\mathrm{d}\,H'}(B)=0$ is impossible according to the first section of the proof of this Theorem. Therefore we have $\mathrm{disp}_{\mathrm{d}\,H'}(B)=1$. There exists a unique integer p' between 1 and d such that $B_{p'}\neq\emptyset$ and $B_{p'}\neq H'g'_{p'}$. We set $B':=B_{p'}$. Then we get $|B'|=1$ as we have seen for A'.

   We have $A'*B'=(Hg_pg'_{p'})\setminus\{e\}$. If $Hg_pg'_{p'}\neq He$, then we have $|H|=|A'*B'|=1$, hence H is the trivial subgroup: a contradiction. Thus $Hg_pg'_{p'}=He$, which implies $|H|=2$ as required.

   If $\mathrm{disp}^{\mathrm{l}}_{\mathrm{H}}(A)=1$ then the same proof may be applied to the quasi-factorization (B,A) by working with the left-cosets of H.

3. notice that H is assumed to normal.

   Since $\mathrm{disp}_{\mathrm{H}}(A)=2$, there exist two distinct cosets Hg_1 and Hg_2 of G such that $\emptyset\subsetneq A\cap Hg_1\subsetneq Hg_1$ and $\emptyset\subsetneq A\cap Hg_2\subsetneq Hg_2$. Let $A_1:=A\cap Hg_1$ and $A_2:=A\cap Hg_2$.

   If there is an H-coset in A then by Lemma 7, A_1 (and A_2) cannot be used twice on the same coset. Thus A_1 is used at least once on a coset distinct from He otherwise we would have $\alpha\leq 1$. Let Hv be this coset. Obviously Hv is not covered with only A_1 because A_1 is not a $H-$coset. Hence A_1 and A_2 are used exactly once to cover Hv. Thus $|Hv|=|A_1|+|A_2|$. Hence $n=0 \pmod{|A_1|+|A_2|}$. If C is any H-coset of A, we have $|C|=|H|=|A_1|+|A_2|$. Thus $\omega=0 \pmod{|A_1|+|A_2|}$. From $n=\alpha\times\omega+1$, we get $n=1 \pmod{|A_1|+|A_2|}$ contradicting $n=0 \pmod{|A_1|+|A_2|}$. Therefore there is no H-coset in A.

   Thus $A=A_1\cup A_2$. As H is a proper subgroup of G, there exists x such that $He\cap Hx=\emptyset$.

   If $|A_1|=|A_2|$, then due to the cover of Hx, we get $n=0 \pmod{|A_1|}$. From $n=\alpha\times\omega+1$, we have $n=1 \pmod{|A_1|}$. Thus $|A_1|=1$. This means that $|A|=2$, which is contradictory to the hypothesis of the Theorem. Hence $|A_1|\neq|A_2|$ and we may assume that $|A_1|>|A_2|$.

   If z is any element of G, let $n_z(A_1)$ (respectively $n_z(A_2)$) be the number of times the tile A_1 (respectively A_2) is used to cover the coset Hz, that is $n_z(A_1)=|\{b\in B|\ A_1b\subseteq Hz\}|$ (respectively $n_z(A_2)=|\{b\in B|\ A_2b\subseteq Hz\}|$). Let $n_{max}(A_1):=\max_{z\in G}\{n_z(A_1)\}$, $n_{min}(A_1):=\min_{z\in G}\{n_z(A_1)\}$, $n_{max}(A_2):=\max_{z\in G}\{n_z(A_2)\}$ and $n_{min}(A_2):=\min_{z\in G}\{n_z(A_2)\}$.

   Fact 6

   $$n_{max}(A_1)=n_{max}(A_2)$$

$$n_{min}(A_1) = n_{min}(A_2)$$

Let z be an element of G such that $n_{max}(A_1) = n_z(A_1)$. Let b be any element of B such that $A_1 b \subseteq Hz$. Then from $A_1 \subseteq Hg_1$, we get $Hz = Hg_1 b$, thus $b \in H{g_1}^{-1}z$ as H is a normal subgroup of G. From $A_2 \subseteq Hg_2$, we get $A_2 b \subseteq Hg_2 H{g_1}^{-1}z = Hg_2{g_1}^{-1}z$. Hence $n_{g_2{g_1}^{-1}z}(A_2) \geq n_{max}(A_1)$. Thus $n_{max}(A_2) \geq n_{max}(A_1)$ and likewise $n_{max}(A_1) \geq n_{max}(A_2)$. The proof establishing the relation $n_{min}(A_1) = n_{min}(A_2)$ is similar.

Let $n_{max} := n_{max}(A_1)$ and $n_{min} := n_{min}(A_1)$.

Fact 7

$$n_{max} > n_{min}$$

If $n_{max} = n_{min}$ then $|Hx| = n_{min} \times (|A_1| + |A_2|)$ and so $n = 0 \pmod{\omega}$, contradicting $n = \alpha \times \omega + 1$.

Fact 8

$$|H| = n_{max} \times |A_1| + n_{min} \times |A_2|$$

.

If g is any element of G, we set $\epsilon(g) = 1$ if $Hg = H$ and we set $\epsilon(g) = 0$ otherwise.

Suppose $n_e(A_1) = n_{max}$. Let z be any element of G such that $n_z(A_2) = n_{min}$. Then we have $|Hz| - 1 \leq n_{max} \times |A_1| + n_{min} \times |A_2| \leq n_{max} \times |A_1| + n_e(A_2) \times |A_2| = |He| - 1$. Thus $n_e(A_2) = n_{min}$ and so $|He| - 1 = n_{max} \times |A_1| + n_{min} \times |A_2|$. There exists g in G such that $n_g(A_1) = n_{min}$ and due to Fact 7, we have $Hg \neq He$. Thus $|Hg| = n_{min} \times |A_1| + n_g \times |A_2| \leq n_{min} \times |A_1| + n_{max} \times |A_2|$. From $|He| - 1 = n_{max} \times |A_1| + n_{min} \times |A_2|$, we get $(n_{max} - n_{min}) \times (|A_1| - |A_2|) \leq -1$, which is impossible.

Hence $n_e(A_1) < n_{max}$. Then there exists g in G such that $n_g(A_1) = n_{max}$ and $Hg \neq He$. If $n_g(A_2) > n_{min}$ then $|Hg| > n_{max} \times |A_1| + n_{min} \times |A_2|$. Let g' be an element in G such that $n_{g'}(A_2) = n_{min}$. Then $|Hg'| - \epsilon(g') \leq n_{max} \times |A_1| + n_{min} \times |A_2| < |Hg|$. Thus $\epsilon(g') = 1$: this means that $Hg' = He$. Then we have $|Hg'| - 1 = n_e(A_1) \times |A_1| + n_{min} \times |A_2|$. Since $n_e(A_1) < n_{max}$, we have $|Hg'| - 1 < n_{max} \times |A_1| + n_{min} \times |A_2| < |H|$, which is impossible: we must have $n_g(A_2) = n_{min}$.

Hence $|Hg| = n_{max} \times |A_1| + n_{min} \times |A_2|$ as required.

Fact 9 *We have $n_{max} = n_{min} + 1$ and $|A_1| = |A_2| + 1$*

Let x be an element of G such that $n_x(A_1) = n_{min}$. Hence we have $|Hx| - \epsilon(x) = n_{min} \times |A_1| + n_x(A_2) \times |A_2|$. Due to Fact 8, we get $\epsilon(x) = (n_{max} - n_{min}) \times |A_1| + (n_{min} - n_x(A_2)) \times |A_2|$. Thus $1 \geq \epsilon(x) \geq (n_{max} - n_{min}) \times (|A_1| - |A_2|)$. As $|A_1| > |A_2|$ and by Fact 7, $n_{max} > n_{min}$, we must have $n_{max} = n_{min} + 1$ and $|A_1| = |A_2| + 1$.

Fact 10 *$\{Hg_1,\ Hg_2\}$ is a partition of G in cosets, thus H is of cardinality $\frac{n}{2}$*

Let Hz be any coset distinct from coset He. By Fact 9, we have $n_z(A_1) = n_{max}$ or n_{min}. If $n_z(A_1) = n_{min}$ then $|Hz| \leq n_{min} \times |A_1| + n_{max} \times |A_2|$. With Fact 8, we get $0 \leq (n_{min} - n_{max}) \times (|A_1| - |A_2|)$, which is impossible. Thus $n_z(A_1) = n_{max}$ and so $n_z(A_2) = n_{min}$. Let b be any element of B such that $A_1 b \subseteq Hz$. Then $A_2 b$ must be a subset of a coset where $n_{max}(A_2)$ is reached. We have seen that for every coset Hz' distinct from He, we have $n_{z'}(A_2) = n_{min}$. Hence $A_2 b$ must lie in He. Thus $Hg_2 b = He$ and $Hg_1 b = Hz$. As H is a normal subgroup of G, we have $Hz = Hg_1{g_2}^{-1}$. Therefore there is exactly one coset distinct from He.

Since Hg_1 and Hg_2 are two cosets, this implies that $\{Hg_1,\ Hg_2\}$ is a partition of G in cosets.

□

Example 4

Let (A, B) be the near-factorization of D_{16} introduced in Example 2: $A = \{e, r^5, sr^5\}$ and $B = \{r, r^2, s, sr, sr^2\}$.

Let $H_1 := \{e, sr^5\}$. Since $\mathrm{disp}_{H_1}(A) = 1$, H_1 must be of cardinal 2.

Let $H_2 := \{e, r, r^2, r^3, r^4, r^5, r^6, r^7\}$. Since $\mathrm{disp}_{H_2}(A) = 2$, $|A| \neq 2$ and H_2 is normal, H_2 must be of cardinal $\frac{16}{2} = 8$.

Theorem 1 may be used to decrease the number of cases to be investigated when looking for a near-factorization for a given group with the help of a computer. From the list of all subsets A of G of cardinal ω, we may keep only those satisfying Theorem 1 and then for every of these A check if there exists a subset B of cardinal α such that (A, B) is a near-factorization. For every group of small order (that is less than 1000), it is quite easy to get the list of all subgroups of G and the list of all normal subgroups of G using *GAP* [9] for instance. Theorem 1 is an interesting filter because it may be applied to any group. Our implementation reveals that it performs quite well when ω or α is small as one might expect. In some groups, there is no symmetric subsets at all satisfying Theorem 1 with the required cardinality. For instance, the only groups of order 16 and of order 40 with a symmetric subset A of cardinal 3 satisfying Theorem 1 are the dihedral groups and cyclic groups.

We will use Theorem 1 to derive Lemma 9 and Lemma 10.

Lemma 8 *If $\omega = 3$, A is symmetric and n is odd then $G(A, B)$ is a web.*

Proof. Since n is odd, there is no involution in G. This implies with $A = A^{-1}$ that there is a in G such that $A = \{a^{-1}, e, a\}$. Let H be the cyclic subgroup generated by a. Notice that $A \subseteq H$, thus $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = \mathrm{disp}^{\mathrm{l}}_{\mathrm{H}}(A) = 1$. If H is distinct from G then by Theorem 1, we must have $|H| = 2$, which is impossible as n is odd. Thus $G = \langle A \rangle$, and so G is a cyclic group. Since $\omega = 3$, $G(A, B)$ is a web [1]. □

András Sebö proved in [13] that the minimal imperfect graphs containing certain configurations of two $\alpha-$critical edges and one co-critical non-edge are exactly the odd holes or anti-holes.

S. Markossian, G. Gasparian, I. Karapetian and A. Markosian also studied in [11] such edges and non-edges in conjunction with the Strong Perfect Graph Conjecture.

Recall that a graph $G(A, B)$ has a co-critical non-edge if and only if $\mathrm{INT}(A) = \omega - 1$. Next Lemma partially characterizes graphs $G(A, B)$ with a co-critical non-edge.

Lemma 9 *Let G be a finite group such that every involution z commutes with every element of G. If (A, B) is a near-factorization of G such that $\mathrm{INT}(A) = \omega - 1$ then G is a cyclic group and $G(A, B)$ is a web.*

Proof.

Since $\mathrm{INT}(A) = \omega - 1$, by Lemma 6 there exists an element y of G such that $|A \cap yA| = \omega - 1$. Let $T := \{e,\ y,\ y^2,\ \ldots,\ y^{|T|-1}\} * t$ be the maximal right-$y-$chain of A. Let H be the cyclic subgroup of G generated by y. Since the right-tiles of A are T and H-right-cosets, we have $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 1$.

Obviously $y \neq e$, hence H is not the trivial subgroup of G. Thus by Theorem 1, we have $H = G$ or $|H| = 2$.

If $|H| = 2$ then y is an involution of G distinct from e, and we must have $|T| = 1$. Hence there must be some H-right-cosets in A. The element y commutes with every element of G, hence H is a normal subgroup of G. If T is used only on the coset $Hu(A, B)$, then $\alpha \leq 1$, which is impossible. Therefore T is used in the cover of another coset Hx. As only T is used on Hx, it is used at least twice, which is in contradiction with Lemma 7 because H is a normal subgroup of G.

Therefore $H = G$, that is G is a cyclic group.

Hence $A = T$ and $G(t^{-1}A, B)$ is a web. Thus $G(A, B)$ which is isomorphic to $G(t^{-1}A, B)$ is a web. □

Lemma 9 is not true if the hypothesis that every involution is in the center of G is not assumed. Indeed the dihedral groups are example of non-cyclic groups having near-factorizations (A, B) and $\mathrm{INT}(A) = \omega - 1$ (see Section 3). Besides we know a graph $G(A, B)$ with 50 vertices such that $\mathrm{INT}(A) = \omega - 1$, which is not a web.

Corollary 1 *If G is a non-cyclic finite abelian group then it admits no near-factorization (A, B) such that $\mathrm{INT}(A) = \omega - 1$.*

Corollary 2 *If G is a non-cyclic finite group of odd order then it admits no near-factorization (A, B) such that $\mathrm{INT}(A) = \omega - 1$.*

Proof. Indeed there is no involution in a group of odd order. □

Example 5

Let G be any group of order $3 \times p + 1$ (p a prime) such that its center contains all its involutions, with a symmetric near-factorization (A, B). We may assume that $|A| = 3$. Since $|A|$ is odd and A is symmetric, there must be an element w in A such that $w^2 = e$. Let a be another element in A. Thus $\{a,\ w\} \subseteq A \cap Aaw$

and so $\mathrm{INT}(A) \geq 2$. *Then by Lemma 9, G must be cyclic. This implies for instance that 7 groups, out of the 14 groups of order 16, have no symmetric near-factorizations.*

There are many non-abelian groups containing in their center all their involutions: according to GAP [9] there are 58 such groups out of the 267 groups of order 64, and 52 such groups out of the 231 groups of order 96. Notice that for $n = 64$ *or* 96, ω *or* α *must be prime, hence any* C*GPW graph of these orders is a web (see Section 2). Thus if any of these groups has a near-factorization* (A, B) *then the graph* $G(A, B)$ *is not a* C*GPW graph. Notice that for* $n = 64$, *these groups do not have any symmetric near-factorization* (A, B) *such that* $|A| = 3$.

Lemma 10 *Let G be a finite group such that all its cyclic subgroups are normal and admitting a near-factorization* (A, B) *such that* $\mathrm{INT}(A) = \omega - 2$. *Then*

- *If G is abelian then G is cyclic.*
- *If G is not abelian then the order of G is a multiple of* 4, *G has an element y of order* $\frac{n}{2}$ *and* $y^{\frac{n}{4}}$ *is the only involution of G .*

Proof. Since $\mathrm{INT}(A) = \omega - 2$, we have $\omega \geq 3$ and there exists an element y of G such that $|A \cap yA| = \omega - 2$. Let $T_1 := \{e,\ y,\ y^2,\ \ldots,\ y^{|T_1|-1}\} * t_1$ and $T_2 := \{e,\ y,\ y^2,\ \ldots,\ y^{|T_2|-1}\} * t_2$ be the two maximal right-$y-$chains of A. Let u be the uncovered element. Let H be the cyclic subgroup generated by the element y. Hence by assumption on G, H is a non-trivial normal subgroup of G:

If $G = H$ then G is abelian and cyclic, thus we are done. Hence we may assume that $H \subsetneq G$.

Since A is made of T_1, T_2 and some H-cosets, we have $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) \leq 2$. By Theorem 1, we have $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) > 0$. If $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 1$ then by Theorem 1 again, we get $|H| = 2$. Since $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 1$, T_1 and T_2 must lie in the same right-coset of H. Thus $T_1 \cup T_2$ is a H-coset in A, and this implies that $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 0$, a contradiction.

Hence $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 2$ and by Theorem 1 again, H has cardinality $\frac{n}{2}$. Therefore y is an element of order $\frac{n}{2}$ and there is no H-coset in A.

Fact 11 *We have* $|T_1| \neq |T_2|$.

Suppose that $|T_1| = |T_2|$. As there is no H-coset in A, we have $|H| = 1 \pmod{|T_1|}$ due to the cover of the coset $Hu(A, B)$. Then we also have $|H| = 0 \pmod{|T_1|}$ due to the cover of the other coset. Hence $|T_1| = 1$. This implies that $|A| = 2$. This is impossible as $\omega \geq 3$.

Thus $|T_1| \neq |T_2|$ and we may assume that $|T_2| < |T_1|$.

Fact 12 *The pair* $\{Ht_1,\ Ht_2\}$ *is a partition of G in right cosets.*

If t_1 and t_2 lie in the same right coset then $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) \leq 1$, contradicting $\mathrm{disp}^{\mathrm{r}}_{\mathrm{H}}(A) = 2$. Thus $Ht_1 \cap Ht_2 = \emptyset$. As $|H| = \frac{n}{2}$, we are done.

Fact 13 *We have* $(Ht_1)^{-1} = Ht_1$ *and* $(Ht_2)^{-1} = Ht_2$.

Suppose that $H = Ht_1$ then we obviously have $(Ht_1)^{-1} = Ht_1$. Since the inversion map is a bijective map, this implies that $(Ht_2)^{-1} = Ht_2$. The proof for the case $H = Ht_2$ is similar.

Fact 14 *If G is abelian then G is a cyclic group.*

If G is abelian then let b be any element of B distinct from $t_2^{-1} * y^{-|T_2|} * u$ that is such that $T_2 b$ is not followed by the uncovered element u. Hence $T_2 b$ is followed by a tile $T_2 b'$ or by a tile $T_1 b'$, that is $t_2 * b' = y^{|T_2|} * t_2 * b$ or $t_1 * b' = y^{|T_2|} * t_2 * b$. Thus $b' = y^{|T_2|} * b$ or $b' = y^{|T_2|} * t_1^{-1} * b$. If $b' = y^{|T_2|} * b$ then $t_1 * b' = t_1 * y^{|T_2|} * b$. Since $|T_2| < |T_1|$, $y^{|T_2|} * t_1$ is an element of T_1. Thus $y^{|T_2|} * t_1$ is an element of A and we have a contradiction. Therefore $b' = y^{|T_2|} * t_1^{-1} * b * t_2$. Let $y' := y^{|T_2|} * t_1^{-1} * t_2$. We have seen that for every element b of B except maybe one, $y'b$ is an element of B. Thus $\mathrm{INT}(B) = \alpha - 1$. Since G is abelian, (B, A) is obviously a near-factorization of G. Hence by Lemma 9, G must be cyclic.

Fact 15 *If G is not abelian then n is a multiple of 4 and* $y^{\frac{n}{4}}$ *is the only involution of G .*

By assumption, G is not abelian.

Let q be an element of G such that $Hq \neq H$.

If n is not a multiple of 4 then $|H|$ is odd. Hence due to Fact 13 there exists at least one element z in Hq such that $z^2 = e$. Since $\langle z \rangle$ is a normal subgroup of G, z must commute with every element of G

and in particular with y. Since z is an element of Hq, there exists an integer i such that $z = y^i * q$. From $z * y = y * z$, we get $y^i * q * y = y^{i+1} * q$. Thus $q * y = y * q$. Due to Fact 12, G must be abelian, which is impossible. Thus n is a multiple of 4 and so $y^{\frac{n}{4}}$ is an involution of G.

Obviously in the coset H there are exactly two involutions: the elements e and $y^{\frac{n}{4}}$. Thus if there is another involution in G then there must be an involution z in Hq, and we have seen that in this case G must be abelian, which is impossible. Hence we are done. □

Corollary 3 *[5] If (A, B) is a near-factorization of a finite abelian group G such that $|A| \leq 4$ then G is cyclic and $G(A, B)$ is a CGPW graph.*

Proof. Let (A, B) be a near-factorization of G such that $|A| \leq 4$. Since G is abelian, we use the additive notation $+$ to denote the operation of G.

If $|A| \leq 3$ then obviously $\mathrm{INT}(A) \geq \omega - 2$. Thus G is cyclic by Lemma 10. Then it is proved in [1] that $G(A, B)$ must be a CGPW graph.

If $|A| = 4$ then n is odd and there is no involution in G. By Lemma 3, there exist x and y in G such that (xA, By) is a symmetric near-factorization. Let $A' := xA$. Since $A' = -A'$ and there is no involution, there are a and a' in G such that $A' = \{a, a', -a, -a'\}$. Then $\{a, a'\} \subseteq A' \cap A' + (a + a')$. Hence $\mathrm{INT}(A') \geq \omega - 2$. By Lemma 10, G must be the cyclic group. Thus $G(A, B) \sim G(A', B')$ is a CGPW graph [1]. □

Example 6

The Quaternion group Q_8 of order 8 is an example of a non-abelian finite group such that all its cyclic subgroups are normal.

There does not seem to be very much non-abelian groups such that all their cyclic subgroups are normal. According to GAP, there is only one (out of 267) such group of order 64: the 262^{th} group. As it has no element of order 32, we know that is has no-near factorization (A, B) such that $|A| = 7$ and $\mathrm{INT}(A) \geq 5$. There is also only one (out of 231) such group of order 96: the 222^{th} group. This group does not have any element of order 48.

In the remaining of this section, we study the problem of characterizing the minimal imperfect graphs in the class of the graphs produced by near-factorizations of finite groups. We first need to recall some results about minimal imperfect graphs.

A *small transversal* is a subset of vertices T such that T is of cardinality at most $\omega + \alpha - 1$ and T meets every maximum clique and every maximum stable set.

In 1976, V. Chvátal found a very useful property of minimal imperfect graphs which states that a minimal imperfect graph contains no small transversal [6].

In 1998, G. Bacsó, E. Boros, V. Gurvich, F. Maffray and M. Preissmann [1] introduced a sufficient condition for partitionable graphs to have a small transversal called the 'Parents Lemma'. A maximum clique K of G is a *mother* of a vertex $x \in K$ if every maximum clique K' containing x satisfies $|K \cap K'| \geq 2$. Similarly, a maximum stable set S of G is a *father* of a vertex $x \in S$ if every maximum stable set S' containing x satisfies $|S \cap S'| \geq 2$.

Lemma 11 'The Parents Lemma' *[1] If a vertex of a partitionable graph has a father and a mother then the graph has a small transversal.*

Then we have the following result:

Lemma 12 *Let G be a finite group of even order such that every involution y commutes with every element of G. If (A, B) is any symmetric near-factorization of G then $G(A, B)$ has a small transversal, hence it is not a counter-example to the Strong Perfect Graph Conjecture.*

Proof. Since n is even, ω and α are necessarily odd.

As ω is odd, there is an element y of A such that $y^2 = e$. We are going to show that A is a mother of y. Let pA be any ω−clique containing y distinct from A. Hence there is a in A such that $y = p * a$. If $a^{-1} = y$ then $p = y * a^{-1} = y^2 = e$ and so $pA = A$, a contradiction. Thus a^{-1} is not equal to y. We have $a^{-1} = y * p = p * y$ because y commutes with p. Thus a^{-1} is an element of $p * A$. Hence $\{a^{-1}, y\} \subset A \cap pA$. This means that A is a mother of y.

Likewise there exists an element x of B such that $B = B^{-1}$ is a father of x. Hence $yx^{-1}B = yx^{-1}B^{-1}$ is a father of y. By applying the Parents Lemma, we see that the graph $G(A, B)$ has a small transversal. □

Corollary 4 *Let G be a finite abelian group of even order. If (A,B) is any near-factorization of G then $G(A,B)$ is not a counter-example to the Strong Perfect Graph Conjecture.*

3 Near-factorizations of the dihedral groups.

In this section, we show how to carry any near-factorization of a cyclic group of even order to the dihedral group of the same order.

We begin by introducing a map ϕ from $\mathbf{Z}_{2n}$ into D_{2n}.

An *even* element of $\mathbf{Z}_{2n}$ is an element of $2\mathbf{Z}_{2n}$. The *odd* elements are the other elements of $\mathbf{Z}_{2n}$. Notice that if x is an even element of $\mathbf{Z}_{2n}$, then there exists a unique integer y between 0 and $(n-1)$ such that $x = 2 \times y$. We denote by $\frac{x}{2}$ this integer.

If x and y are two even elements of $\mathbf{Z}_{2n}$ then we have $\frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} \pmod{n}$ and if x is any element of $\mathbf{Z}_{2n}$ then we have $\frac{2x}{2} = x \pmod{n}$.

Let ϕ be the bijective map of $\mathbf{Z}_{2n}$ onto D_{2n} defined by:

$$\begin{array}{rccc} \phi: & \mathbf{Z}_{2n} & \to & D_{2n} \\ & \text{x is even} & \mapsto & r^{\frac{x}{2}} \\ & \text{x is odd} & \mapsto & sr^{\frac{x-1}{2}} \end{array}$$

We now state some properties of ϕ which are useful for the proofs:

Lemma 13 *For all x and y of $\mathbf{Z}_{2n}$ such that y is even, we have $\phi(x+y) = \phi(x) * \phi(y)$.*

For all x and y of $\mathbf{Z}_{2n}$, we have:

- *if y is even, $\phi(x) * \phi(y)^{-1} = \phi(x-y)$.*
- *if y is odd, $\phi(x) * \phi(y)^{-1} = \phi(y-x)$.*

Proof.

If x and y are even then we have $\phi(x+y) = r^{\frac{x+y}{2}} = r^{\frac{x}{2}+\frac{y}{2}} = r^{\frac{x}{2}} * r^{\frac{y}{2}} = \phi(x) * \phi(y)$ and $\phi(x-y) = r^{\frac{x-y}{2}} = r^{\frac{x}{2}+\frac{-y}{2}} = r^{\frac{x}{2}} * r^{\frac{-y}{2}} = \phi(x) * \phi(y)^{-1}$.

If x is odd and y is even then we have $\phi(x+y) = sr^{\frac{x+y-1}{2}} = sr^{\frac{x-1}{2}+\frac{y}{2}} = sr^{\frac{x-1}{2}} * r^{\frac{y}{2}} = \phi(x) * \phi(y)$ and $\phi(x-y) = sr^{\frac{x-y-1}{2}} = sr^{\frac{x-1}{2}+\frac{y}{2}} = sr^{\frac{x-1}{2}} * r^{\frac{-y}{2}} = \phi(x) * \phi(y)^{-1}$.

Hence, if y is even then we have $\phi(x+y) = \phi(x) * \phi(y)$ and $\phi(x) * \phi(y)^{-1} = \phi(x-y)$.

If x is even and y is odd then we have $\phi(x) * \phi(y)^{-1} = r^{\frac{x}{2}} * sr^{\frac{y-1}{2}} = sr^{\frac{y-x-1}{2}} = \phi(y-x)$.

If x is odd and y is odd then we have $\phi(x) * \phi(y)^{-1} = sr^{\frac{x-1}{2}} * sr^{\frac{y-1}{2}} = r^{\frac{y-x}{2}} = \phi(y-x)$.

Hence, if y is odd then we have $\phi(x) * \phi(y)^{-1} = \phi(y-x)$. □

From a near-factorization (A,B) of $\mathbf{Z}_{2n}$, we get a near-factorization of D_{2n} this way:

Algorithm 1 Carrying a near-factorization of $\mathbf{Z}_{2n}$ into D_{2n}

Input: a near-factorization (A,B) of $\mathbf{Z}_{2n}$

Output: a near-factorization (A',B') of D_{2n}

Step 1: find x an element of $\mathbf{Z}_{2n}$ such that $A+x$ is symmetric and let $A_1 := A+x$ (exists by Lemma 3).

Step 2: take a_1 an element of A_1 and let $A_2 := A_1 + a_1$.

Step 3: let B_0 be the set of the even elements of B and B_1 be the set of the odd elements of B. Then take $A' := \phi(A_2)$ and $B' := \phi(B_0) \cup \phi(B_1)r^{a_1}$.

We say that (A',B') is a *dihedral near-factorization associated to* (A,B).

Obviously one may get several distinct near-factorizations of D_{2n} through this algorithm from one near-factorization of $\mathbf{Z}_{2n}$ as x is not uniquely defined in `Step 1` and neither is a_1 in `Step 2`.

We first prove that any couple (A',B') produced by this algorithm is indeed a near-factorization of D_{2n}.

Theorem 2 *Let (A,B) be a near-factorization of $\mathbf{Z}_{2n}$. Let (A',B') be an output of algorithm 1 with input (A,B). Then (A',B') is a near-factorization of D_{2n}.*

Proof. Recall that due to the algorithm, we have $A' = \phi(A_2)$ and $A_2 = A_1 + a_1$ where A_1 is symmetric and a_1 is an element of A_1.

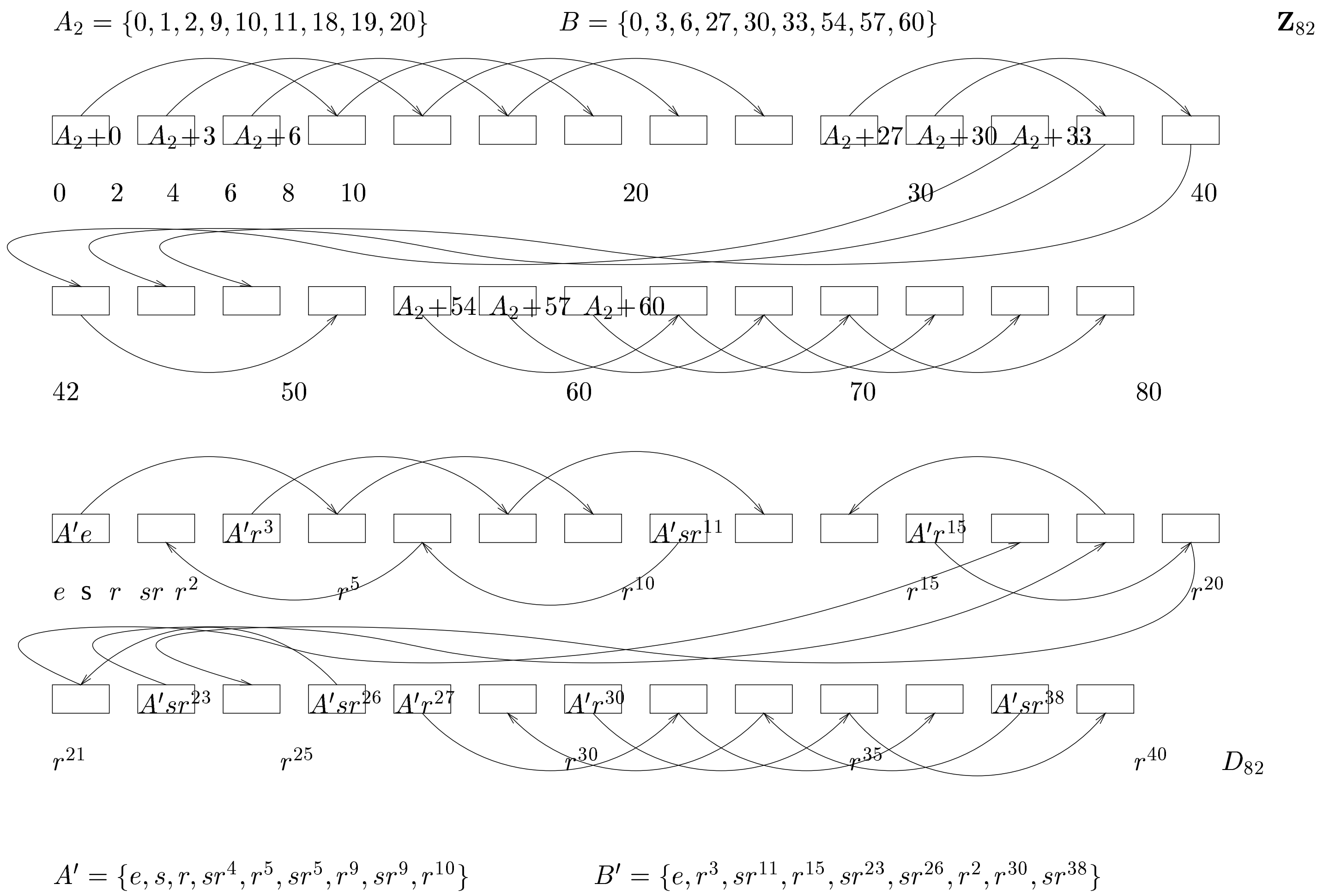


Figure 3: The couple (A', B') is a near-factorization of D_{82} induced by the near-factorization (A_2, B) of $\mathbf{Z}_{82}$

Fact 16 *For every b of B, there exists b' in B' such that $\phi(A_2 + b) = A'b'$.*

If b is even then let a be any element of A_2. By Lemma 13, we have $\phi(a + b) = \phi(a) * \phi(b)$. Hence $\phi(A_2 + b) \subseteq \phi(A_2)\phi(b)$. Since ϕ is a bijective map, we get $\phi(A_2 + b) = \phi(A_2)\phi(b)$ with $\phi(b) \in B'$. Thus we are done.

If b is odd then let a be any element of A_2. By definition of A_2, $a - a_1$ is an element of A_1, which is a symmetric set. Hence $a_1 - a$ is an element of A_1. Thus $2a_1 - a$ is an element of A_2. Notice that $2a_1 + b$ is odd. Let $b' := \phi(2a_1 + b)$. As $\phi(2a_1 + b) = sr^{a_1 + \frac{b-1}{2}} = sr^{\frac{b-1}{2}} * r^{a_1}$, b' is an element of B'. If a is even then $\phi(2a_1 - a) * b' = r^{a_1 - \frac{a}{2}} * sr^{a_1 + \frac{b-1}{2}} = sr^{\frac{a+b-1}{2}} = \phi(a + b)$. Hence $\phi(a + b) \in A'b'$. If a is odd then $\phi(2a_1 - a) * b' = sr^{\frac{2a_1 - a - 1}{2}} * sr^{\frac{2a_1 + b - 1}{2}} = r^{\frac{a+b}{2}} = \phi(a + b)$. Thus $\phi(a + b) \in A'b'$. Therefore we have $\phi(A_2 + b) \subseteq A'b'$. This implies that $\phi(A_2 + b) = A'b'$ because ϕ is a bijective map.

Fact 17 *The couple (A', B') is a near-factorization of D_{2n}.*

We have seen that $\{\phi(A_2 + b),\ b \in B\} \subseteq \{A'b',\ b' \in B'\}$. Since ϕ is a bijective map, there exists u in D_{2n} such that $\{\phi(A_2 + b),\ b \in B\}$ is a partition of $D_{2n} \setminus \{u\}$. As B and B' are of equal cardinality, we get that $\{A'b',\ b' \in B'\}$ is a partition of $D_{2n} \setminus \{u\}$. Therefore (A', B') is a near-factorization of D_{2n}. $\square$

We now prove that the graph $G(A', B')$ is not altered by the choice of x in `Step 2` or by the choice of a_1 in `Step 3`.

Lemma 14 *Let (A, B) be a near-factorization of Z_{2n}. Let (A', B') and (A'', B'') be two dihedral near-factorization associated to (A, B). Then the graph $G(A', B')$ is isomorphic to the graph $G(A'', B'')$.*

Proof. By construction, there exist two elements x and y of $\mathbf{Z}_{2n}$ such that $A' = \phi(A + x)$ and $A'' = \phi(A + y)$.

We have

$$\begin{aligned} A' &= \phi(A+x) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A+x\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A+x\} \end{aligned}$$

and

$$\begin{aligned} A'' &= \phi(A+y) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A+y\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A+y\} \end{aligned}$$

If $y-x$ is even then by taking the unique integer j between 0 and $n-1$ such that $2j = 2i+x-y \pmod{2n}$, we get

$$\begin{aligned} A'' &= \left\{r^{j+\frac{y-x}{2}} \mid 0 \le j \le n-1,\ 2j \pmod{2n} \in A+x\right\} \\ &\quad \bigcup \left\{sr^{j+\frac{y-x}{2}} \mid 0 \le j \le n-1,\ 2j+1 \pmod{2n} \in A+x\right\} \end{aligned}$$

Hence, $A'' = A' r^{\frac{y-x}{2}}$. Thus we have ${A''}^{-1}A'' = r^{-\frac{y-x}{2}} {A'}^{-1} A' r^{\frac{y-x}{2}}$. This means that the connecting set $({A''}^{-1}A'') \setminus \{e\}$ is the image of $({A'}^{-1}A') \setminus \{e\}$ under the inner automorphism $g \mapsto r^{-\frac{y-x}{2}} g r^{\frac{y-x}{2}}$. Then Lemma 4 implies that the Cayley graph $G(A'', B'')$ is isomorphic to the Cayley graph $G(A', B')$.

The case $y-x$ is odd is slightly trickier.

Let k be an element of $\mathbf{Z}_{2n}$ such that $A+k$ is symmetric. Let $A_{\text{sym}} := A+k$. We have $A' = \phi(A_{\text{sym}} + (x-k))$ and $A'' = \phi(A_{\text{sym}} + (y-k))$. Thus

$$\begin{aligned} A' &= \phi(A_{\text{sym}} + (x-k)) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (x-k)\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}} + (x-k)\} \end{aligned}$$

and

$$\begin{aligned} A'' &= \phi(A_{\text{sym}} + (y-k)) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (y-k)\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}} + (y-k)\} \end{aligned}$$

For every integer p between 0 and $n-1$, we have:

$$\begin{aligned} A' sr^p &= \left\{sr^{p-i} \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (x-k)\right\} \\ &\quad \bigcup \left\{r^{p-i} \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}} + (x-k)\right\} \\ &= \left\{sr^{p+i} \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (k-x)\right\} \\ &\quad \bigcup \left\{r^{p+i} \mid 0 \le i \le n-1,\ 2i-1 \pmod{2n} \in A_{\text{sym}} + (k-x)\right\} \\ &= \left\{sr^{p+i} \mid 0 \le i \le n-1,\ 2i+x-2k+y \pmod{2n} \in A_{\text{sym}} + (y-k)\right\} \\ &\quad \bigcup \left\{r^{p+i} \mid 0 \le i \le n-1,\ 2i-1+x-2k+y \pmod{2n} \in A_{\text{sym}} + (y-k)\right\} \end{aligned}$$

Thus by taking $p = -k + \frac{(y+x)-1}{2} \pmod{n}$, we have $A' sr^p = A''$. Hence ${A''}^{-1}A'' = sr^p {A'}^{-1} A' sr^p$. Therefore the connecting set $({A''}^{-1}A'') \setminus \{e\}$ is the image of $({A'}^{-1}A') \setminus \{e\}$ under the inner automorphism $g \mapsto sr^p g sr^p$. This implies that the Cayley graph $G(A'', B'')$ is isomorphic to the Cayley graph $G(A', B')$. □

Thus from a near-factorization (A, B) of $\mathbf{Z}_{2n}$, we get a unique partitionable graph $G(A', B')$ where (A', B') is any dihedral near-factorization associated to (A, B). It remains to know if we may get some 'new' partitionable graphs this way. We have not succeeded in proving that in general the graph $G(A', B')$ is isomorphic to $G(A, B)$ when (A, B) is any near-factorization of the cyclic group.

Nevertheless, in Theorem 3 we prove that this is true for all the graphs $G(A, B)$ which are known so far. Before proceeding to the proof, we need to recall how these graphs are constructed.

In 1979, V. Chvátal, R.L. Graham, A.F. Perold and S.H. Whitesides [7] introduced a method to produce a large class of near-factorizations of the cyclic groups $\mathbf{Z}_n$.

Two subsets A_1 and B_1 of $\mathbf{N}$ are said to form a *near-factorization in integers* if and only if $A_1 + B_1 = [0..(|A_1| \times |B_1| - 1)]$. Obviously, a near-factorization in integers induces a near-factorization of $\mathbf{Z}_{|A_1| \times |B_1| + 1}$.

Let (A_1, B_1) be a near-factorization in integers such that $A_1 + B_1 = [0..n_1 - 2]$. Let k, k' be any positive integers.

One may obtain a near-factorization in integers (A_2, B_2) such that $A_2 + B_2 = [0..n_2 - 2]$ with

$$n_2 := (|A_1| \times k) \times (|B_1| \times k') + 1$$

by defining:

$$A_2 := A_1 + (n_1 - 1) \times [0..k - 1] \textit{ and } B_2 := B_1 + (n_1 - 1) \times k \times [0..k' - 1]$$

A *CGPW* graph is a graph $G(A, B)$ where (A, B) is obtained with a finite number of applications of this method starting from a *basic factorization*, that is a near-factorization (A_1, B_1) such that $A_1 = [0..|A_1| - 1]$ and $B_1 = |A_1| \times [0..|B_1| - 1]$.

Explicitly, the CGPW graph G given by $2p$ positive integers $a_1, \ldots, a_{2p}$ is constructed in this way:

- take $A_1 = [0..a_1 - 1]$ and $B_1 = a_1 \times [0..a_2 - 1]$. Set $n_1 = a_1 \times a_2 + 1$.
- take $k = a_3$ and $k' = a_4$ then calculate A_2 and B_2. Set $n_2 = a_1 \times a_2 \times a_3 \times a_4 + 1$.
- take $k = a_5$ and $k' = a_6$ then calculate A_3 and B_3 starting from A_2 and B_2. Set $n_3 = a_1 \times a_2 \times a_3 \times a_4 \times a_5 \times a_6 + 1$.
- $\ldots$
- until $k = a_{2p-1}$ and $k' = a_{2p}$.

G is $G(A_p, B_p)$ and is denoted by $C[a_1, \ldots, a_{2p}]$. By construction, $|A_p| = a_1 \times a_3 \times \ldots \times a_{2p-1} = \omega$, $|B_p| = a_2 \times a_4 \times \ldots \times a_{2p} = \alpha$ and $n_p = a_1 \times a_2 \times \ldots \times a_{2p} + 1 = \alpha \times \omega + 1$.

Following [1], a near-factorization produced by this method is called a *De Bruijn near-factorization*. We call *De Bruijn dihedral near-factorization* any dihedral near-factorization associated to a De Bruijn near-factorization.

Grinstead conjectured in 1984 that every circular partitionable graph is a CGPW graph. This conjecture is still open though it is proved for some small values of α or ω in [1].

Next Theorem shows that the De Bruijn dihedral near-factorizations do not produce any 'new' partitionable graph.

Theorem 3 *If (A, B) is a De Bruijn near-factorization of $\mathbf{Z}_{2n}$ then the graph $G(A, B)$ is isomorphic to the graph $G(A', B')$ where (A', B') is a dihedral near-factorization associated to (A, B).*

Proof.

We first calculate a dihedral near-factorization (A', B') associated to (A, B). Notice that due to Lemma 14, we may proceed without having to fear any loss of generality.

Let $a_1, \cdots, a_{2p}$ be the parameters of the graph $G(A, B)$,that is such that $G(A, B) = C[a_1, \cdots, a_{2p}]$. As $2n$ is even, $|A|$ and $|B|$ must be odd. This implies that the $2p$ parameters a_i are oll odd and that the p parameters $n_1, \cdots, n_p$ are all even. We set $n_0 := 2$ in order to avoid a special case in the proof.

Let $a^+ := (a_1 - 1) + \sum_{k=1}^{p-1} \left(\Pi_{i=1}^{2k} a_i\right)(a_{2k+1} - 1)$. Notice that a^+ is the greatest element of A seen as a set of integers and that it is an even element of A such that $A - \frac{a^+}{2}$ is symmetric. Thus in `Step 1`, we may take $x = -\frac{a^+}{2}$.

Since x is an element of $A - \frac{a^+}{2}$, we may take $A_2 := A$ in `Step 2`. Hence by taking $A' := \phi(A)$ and B' as defined in `Step 3`, we get a dihedral near-factorization associated to (A, B).

Fact 18 *We have $A' * A'^{-1} = \phi(A - A)$.*

We have to prove that $\phi(A) * \phi(A)^{-1} = \phi(A - A)$.

We first prove the inclusion $\phi(A) * \phi(A)^{-1} \subseteq \phi(A - A)$. Let w be any element of $\phi(A) * \phi(A)^{-1}$: there exist a and a' in A such that $w = \phi(a) * \phi(a')^{-1}$. Hence by Lemma 13, we have $w = \phi(a - a')$ or $\phi(a' - a)$. In both cases, w is an element of $\phi(A - A)$. Thus $\phi(A) * \phi(A)^{-1} \subseteq \phi(A - A)$.

We now prove the converse inclusion. Let w be any element of $\phi(A - A)$; there exist a and a' in A such that $w = \phi(a' - a)$.

If a' is even then $w = \phi(a) * \phi(a')^{-1}$ hence it is an element of $\phi(A) * \phi(A)^{-1}$.

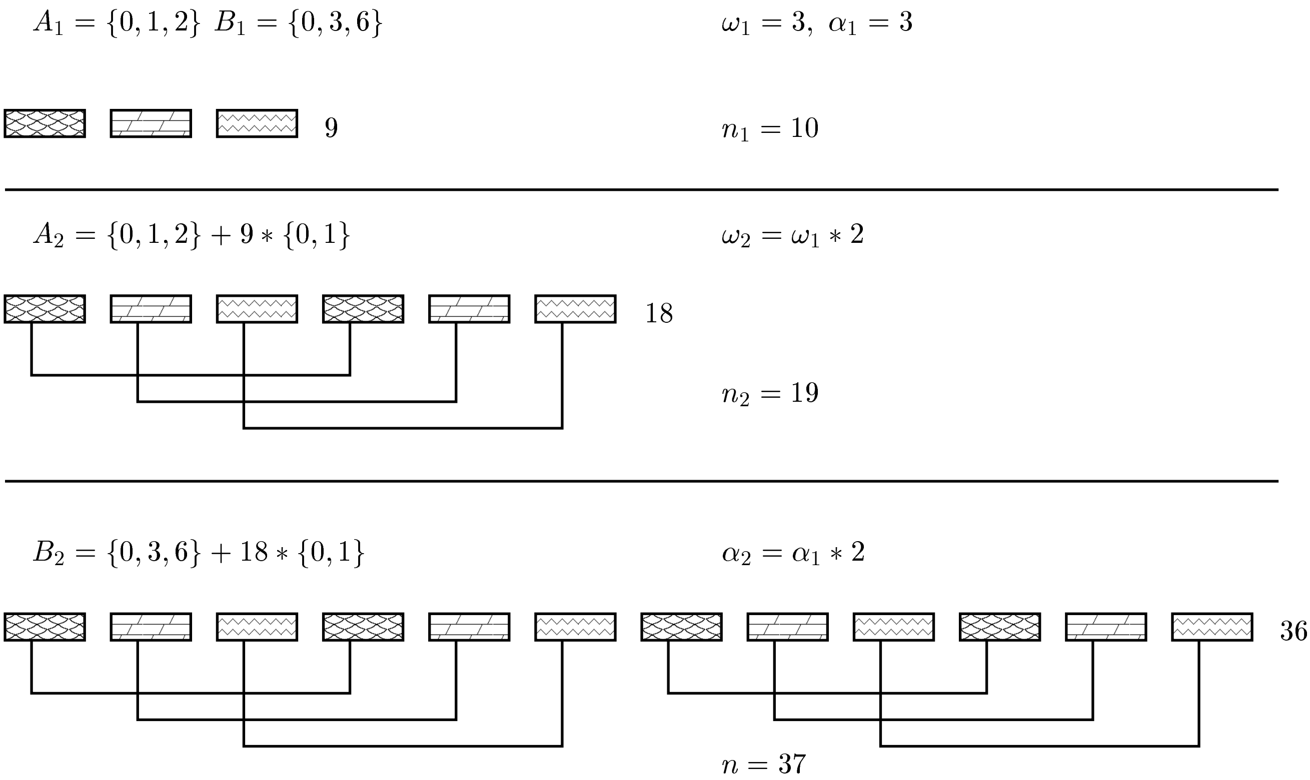


Figure 4: The De Bruijn near-factorization given by $a_1 = 3,\ a_2 = 3,\ a_3 = 2,\ a_4 = 1,\ a_5 = 1,\ a_6 = 2$

If a' is odd, then due to the definition of A, there exist integers $\delta_0,\ \delta_1,\ \ldots, \delta_{p-1}$ and $\delta'_0,\ \delta'_1,\ \ldots, \delta'_{p-1}$ such that $a = \delta_0 + (n_1 - 1)\delta_1 + \ldots + (n_p - 1)\delta_{p-1}$ and $a' = \delta'_0 + (n_1 - 1)\delta'_1 + \ \ldots + (n_p - 1)\delta'_{p-1}$ with $0 \leq \delta_i,\ \delta'_i \leq (a_{2i+1} - 1)$ for every i between 0 and $p-1$. Since a' is odd, there must be an integer k between 0 and $p-1$ such that $0 < \delta'_k < (a_{2k+1} - 1)$ because all the $a_{2i+1} - 1$ are even. Thus $a_{2k+1} > 1$.

If $\delta_k = 0$ then $a + (n_k - 1)$ is an element of A and $a' + (n_k - 1)$ is an element of A. Then $w = \phi(a - a') = \phi((a + n_k - 1) - (a' + n_k - 1)) = \phi(a + n_k - 1) * \phi(a' + n_k - 1)^{-1}$ because $a' + n_k - 1$ is even as $n_k = a_1 * a_2 * a_3 * \ldots * a_{2k} + 1$ is even. Therefore w is an element of $\phi(A) * \phi(A)^{-1}$.

If $\delta_k > 0$ then $a - (n_k - 1)$ is an element of A and $a' - (n_k - 1)$ is an element of A. Then $w = \phi(a - a') = \phi((a - n_k + 1) - (a' - n_k + 1)) = \phi(a - n_k + 1) * \phi(a' - n_k + 1)^{-1}$ because $a' - n_k + 1$ is even. Hence w is an element of $\phi(A) * \phi(A)^{-1}$.

Thus $\phi(A - A) \subseteq \phi(A) * \phi(A)^{-1}$.

Therefore $\phi(A - A) = \phi(A) * \phi(A)^{-1}$.

Fact 19 *Let Γ be the graph with vertex set D_{2n} and with edge set $\{\{x,y\},\ x * y^{-1} \in (A' * A'^{-1}) \setminus \{e\}\}$. Then $G(A,B)$ is isomorphic to Γ.*

Let $\{i,j\}$ be any edge of $G(A,B)$. Then $i - j \in (A - A) \setminus \{0\}$. Thus $j - i \in (A - A) \setminus \{0\}$. Hence $\phi(i - j) \in \phi((A - A) \setminus \{0\})$ and $\phi(j - i) \in \phi((A - A) \setminus \{0\})$. Thus $\phi(i)\phi(j)^{-1} \in \phi((A - A) \setminus \{0\})$. So $\phi(i)\phi(j)^{-1} \in (\phi(A)\phi(A)^{-1}) \setminus \{e\}$. Therefore $\{\phi(i), \phi(j)\}$ is an edge of Γ.

Let $\{\phi(i), \phi(j)\}$ be any edge of Γ. Then $\phi(i)\phi(j)^{-1} \in (\phi(A)\phi(A)^{-1}) \setminus \{e\}$. Since $\phi(i)\phi(j)^{-1}$ is equal to $\phi(i - j)$ or $\phi(j - i)$, we get $\phi(i - j) \in \phi((A - A) \setminus \{0\})$ or $\phi(j - i) \in \phi((A - A) \setminus \{0\})$. Hence $i - j \in (A - A) \setminus \{0\}$, that is $\{i,j\}$ is an edge of $G(A,B)$.

Fact 20 *There exists an element g such that gA' is a symmetric subset of D_{2n}*

Let k be an element of $\mathbf{Z}_{2n}$ such that $A + k$ is a symmetric subset of $\mathbf{Z}_{2n}$.

Let A_0 be the set of the even elements of A and let A_1 be the set of the odd elements of A. Let H be the subgroup of D_{2n} generated by r.

If k is even then $r^{\frac{k}{2}}A' = r^{\frac{k}{2}}\phi(A) = r^{\frac{k}{2}}\phi(A_0) \cup r^{\frac{k}{2}}\phi(A_1) = \phi(A_0 + k) \cup r^{\frac{k}{2}}\phi(A_1)$. Then $r^{\frac{k}{2}}\phi(A_1)$ is a subset of sH, thus it is a symmetric subset of D_{2n} as every of its elements is an involution. The set $\phi(A_0 + k)$ is a symmetric subset of D_{2n} because $A_0 + k$ is a symmetric subset of $\mathbf{Z}_{2n}$. Hence $r^{\frac{k}{2}}A'$ is symmetric.

If k is odd then $sr^{-\frac{k+1}{2}}A' = sr^{-\frac{k+1}{2}}\phi(A_0) \cup sr^{-\frac{k+1}{2}}\phi(A_1)$. The set $sr^{-\frac{k+1}{2}}\phi(A_0)$ is a symmetric subset of D_{2n} as it is a subset of sH. We have $\phi(A+k) = sr^{\frac{k-1}{2}}\phi(A_0) \cup sr^{-\frac{k+1}{2}}\phi(A_1)$, hence $sr^{-\frac{k+1}{2}}\phi(A_1) = H \cap \phi(A+k) = \phi(A_1+k)$. Since A_1+k is a symmetric subset of $2\mathbf{Z}_n$, this implies that $\phi(A_1+k)$ is symmetric, thus $sr^{-\frac{k+1}{2}}\phi(A_1)$ is symmetric. Therefore $sr^{-\frac{k+1}{2}}A'$ is symmetric.

Fact 21 *The graph $G(A', B')$ is isomorphic to the graph $G(A, B)$.*

All we have to show is that $G(A', B')$ is isomorphic to Γ.

Let g be an element of D_{2n} such that gA' is symmetric and let $A'' := gA'$.

Obviously, $G(A', B')$ is isomorphic to $G(A'', B')$. Let Γ' be the graph with edge set D_{2n} and with edge set $\{\{x, y\},\ x * y^{-1} \in (A'' * A''^{-1}) \setminus \{e\}\}$.

Let `inv` be the bijective map of D_{2n} onto itself which maps an element onto its inverse. $\{x, y\}$ is an edge of $G(A'', B')$ if and only if $x^{-1}y \in (A''^{-1}A'') \setminus \{e\}$, that is if and only if $\mathrm{inv}(x)\mathrm{inv}(y)^{-1} \in (A''A''^{-1}) \setminus \{e\}$ as $A'' = A''^{-1}$, hence if and only if $\{\mathrm{inv}(x),\ \mathrm{inv}(y)\}$ is an edge of Γ'. Hence $G(A'', B')$ is isomorphic to Γ'.

Let h denote the inner automorphism of D_{2n} which maps an element x onto $g^{-1}xg$. Then $\{x, y\}$ is an edge of Γ' if and only if $\{h(x), h(y)\}$ is an edge of Γ. Thus Γ' is isomorphic to Γ.

Therefore $G(A', B')$ is isomorphic to Γ. $\square$

In 1990, D. De Caen, D.A. Gregory, I.G. Hughes and D.L. Kreher [5] described a class of near-factorizations of the dihedral groups: if ω is any divisor of $2n-1$, then let $\alpha := \frac{2n-1}{\omega}$ and define

$$A := \left\{r^i,\ 1 \leq i \leq \frac{\omega-1}{2}\right\} \cup \left\{sr^i,\ 0 \leq i \leq \frac{\omega-1}{2}\right\}$$

$$B := \left\{r^{i\omega},\ 0 \leq i \leq \frac{\alpha-1}{2}\right\} \cup \left\{sr^{i\omega},\ 1 \leq i \leq \frac{\alpha-1}{2}\right\}$$

The graph associated to these near-factorizations are a strict subset of the CGPW graphs of even order:

Lemma 15 *The graphs $G(A, B)$ produced by this method are webs.*

Proof. We have $A = \left\{s, r, sr, r^2, sr^2, \cdots, r^{\frac{\omega-1}{2}}, sr^{\frac{\omega-1}{2}}\right\}$.

Consider the De Bruijn near-factorization of $\mathbf{Z}_{2n}$ given by $A_0 := \{0, 1, \cdots, \omega - 1\}$ and by $B_0 := \omega * \{0, \cdots, \alpha - 1\}$. Let $A' := \phi(A_0)$. We know that there exists B' such that (A', B') is a near-factorization of D_{2n} with $G(A', B')$ isomorphic to $G(A_0, B_0)$. We have $A' = \left\{e, s, r, \cdots, r^{\frac{\omega-1}{2}}\right\}$. Thus $A' = Asr^{\frac{\omega-1}{2}}$. Hence $A'^{-1}A' = sr^{\frac{\omega-1}{2}}A^{-1}Asr^{\frac{\omega-1}{2}}$. This means that the connection set of $G(A, B)$ is the image under an inner automorphism of D_{2n} of the connection set of $G(A', B')$. Thus $G(A, B)$ is isomorphic to $G(A', B')$. As $G(A', B')$ is isomorphic to $G(A_0, B_0)$ which is a web, we are done. $\square$

4 Some open questions

This paper gives rise to several questions. We first recall Grinstead's Conjecture:

Conjecture 1 *If (A, B) is a near-factorization of the cyclic group $\mathbf{Z}_n$ then there exists a De Bruijn near-factorization (A', B') such that $G(A, B)$ is isomorphic to $G(A', B')$.*

Grinstead has verified by computer this conjecture for groups of order less than 49, and Bacsó, Boros, Gurvich, Maffray and Preissmann have proved it when A is of cardinal at most 5.

We do not know any near-factorization (A, B) of the dihedral groups whose associated graph $G(A, B)$ is not a CGPW graph. Thus we ask this question, which may be seen as Grinstead's conjecture in dihedral groups:

Problem 1 *If (A, B) is a near-factorization of the dihedral group $\mathbf{D}_{2n}$, is $G(A, B)$ always isomorphic to a graph $G(A', B')$ with (A', B') a De Bruijn dihedral near-factorization ?*

We believe that this is not true because in a dihedral group, a tile may be used 'backwards' (see figure 3), which is not possible in the cyclic group. Hence a tiling of $D_{2n} \setminus \{u\}$ does not behave in the same way than a tiling of $\mathbf{Z}_{2n} \setminus \{u\}$, whereas a positive answer to Problem 1 would suggest the opposite.

With the help of Theorem 1, an exhaustive search by computer revealed that the only groups of order strictly less than 64 having a symmetric near-factorization are the cyclic groups and the dihedral groups. Hence this leads to this natural question:

Problem 2 *Are the cyclic groups and the dihedral groups the only groups having symmetric near-factorizations ?*

Recently, Boros, Gurvich and Hougardy [4] introduced a construction of partitionable graphs generalizing the first construction of Chvátal, Graham, Perold and Whitesides. Let us call BGH$-$graphs the partitionable graphs produced by this new method. All the BGH$-$graphs contain a critical ω-clique, that is an $\omega-$clique which intersects only $2\omega - 2$ other $\omega-$cliques.

Our computer experiments revealed that the third group of order 50 (under GAP classification) has a near-factorization (A, B) such that the graph $G(A, B)$ does not have any critical ω-clique. Thus this graph is not a BGH$-$graph, and neither is it a C$GPW-$graph. Hence near-factorizations of finite groups do produce 'new' partitionable graphs.

Problem 3 *Is it possible to describe a class of near-factorizations of a sequence of finite groups, whose associated graphs are 'new' partitionable graphs?*

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