UNIVERSITE D'ORLEANS
Faculté des Sciences
LIFO

Laboratoire d'Informatique Fondamentale d'Orléans
4, rue Léonard de Vinci, BP 6759
F-45067 Orléans Cedex 2
FRANCE

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A condition for a graph to contain k independent edges

Adam Paweł Wojda
Wydział Matematyki Stosowanej AGH, Kraków, Poland
and
Université d'Orléans, LIFO

Rapport N^o **2000-13**

A condition for a graph to contain k independent edges

A. Paweł Wojda*
Wydział Matematyki Stosowanej AGH, Kraków, Poland
and
LIFO, Université d'Orléans, France
e-mail: wojda@uci.agh.edu.pl, wojda@lifo.univ-orleans.fr

Abstract

Let k, l, n be nonnegative integers such that $1 \le k \le n/2$, and let G be a graph of order n with the minimum vertex-degree $\delta(G) \ge l$. We prove that if the size $e(G)$ of G verifies $e(G) > F(n,k,l) = \max\{f(n,k,l), f(n,k,k-1)\}$, where $f(n,k,l) = \binom{2k-l-1}{2} + l(n-2k+l+1)$ then G contains kK_2. Moreover, if $e(G) = F(n,k,l)$ and G contains no kK_2 then $l \le k-1$ and $G = K_{2k-2p-1} * K_p * \bar{K}_{n-2k+p+1}$, where $p \in \{l, k-1\}$.
We conjecture a similar statement for forests with at most k edges.

1 Introduction

For graphs G and H (which will be supposed to have disjoint vertex sets) we denote by $G * H$ the graph with $V(G * H) = V(G) \cup V(H)$ and $E(G * H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. $\bar{K}_p$ stands for the complement of the complete graph K_p, i.e. the graph with p vertices and no edges. The k-matching or kK_2 is the graph of order $2k$ and size k having k independent edges.
Let us define the following function

$$g(n,k) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

In 1959 Erdös and Gallai [4] proved that every graph G of order n and size $e(G) \ge g(n,k)$ contains a k-matching unless $G = K_{2k-1} \cup \bar{K}_{n-2k+1}$ or $G = K_1 * K_{k-1} * \bar{K}_{n-k}$ (the graphs $G(n,k,0)$ and $G(n,k,k-1)$, respectively, shown in Fig. 1). In 1963

*Partially supported by KBN grant 2 P03A 016 18

Erdös and Sós conjectured (see [3]) that every graph of order n and size greater than $g(n,k)$ contains every forest with at most k edges. This conjecture has been proved thirty years later by Brandt [1].

Theorem 1 (Brandt) *Suppose G is a graph of order n and size $e(G) > g(n,k)$. Then G contains every forest on k edges without isolated vertices.*

In the second section we give a sufficient condition on minimum vertex degree and size of a graph to contain a k-matching, and we conjecture a corresponding statement for forests with at most k edges.
Conditions on minimum vertex degree and size of graphs were considered for many properties of graphs. May be the first result of this type is the following, proved by Paul Erdös in [2] for hamiltonian graphs.

Theorem 2 (Erdös) *Let G be a graph on n vertices and k a natural number, $1 \le k \le n/2$. If $\delta(G) \ge k$ and the size of G*

$$e(G) > \max\left\{ \binom{n-k}{2} + k^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \lfloor \frac{n-1}{2} \rfloor^2 \right\}$$

then G is hamiltonian.

2 Result and conjecture

For given nonnegative integers n, k and l, such that $1 \le k \le n/2$, $0 \le l \le k-1$, let us define the graph $G(n,k,l) = K_{2k-2l-1} * K_l * \bar{K}_{n-2k+l+1}$ (see Figure 1).

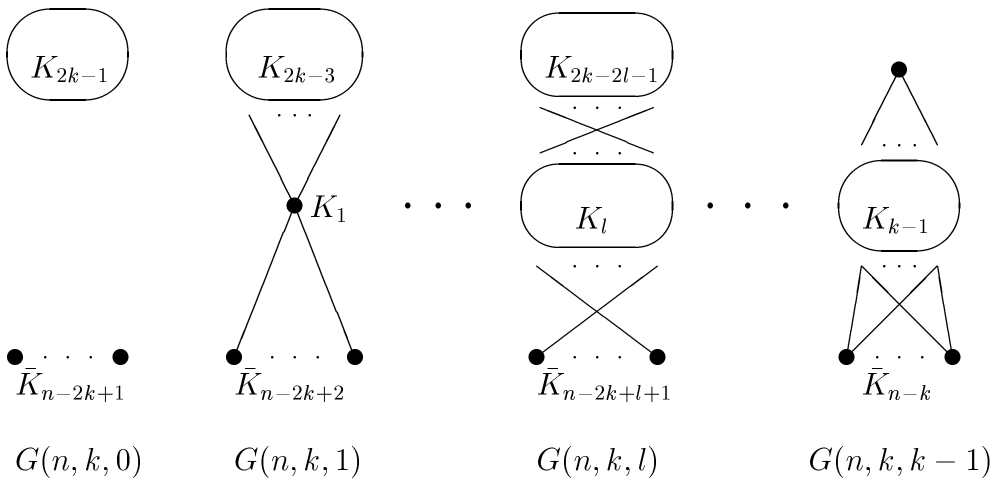

Fig. 1

Note that $\delta(G(n,k,l)) = l$ and $G(n,k,l)$ does not contain any k-matching. Denote the size of $G(n,k,l)$ by $f(n,k,l)$ and let

$$F(n,k,l) = \max\{f(n,k,l), f(n,k,k-1)\}$$

We shall prove the following result.

Theorem 3 *If G is a graph of order n with $\delta(G) \geq l$ and $e(G) \geq F(n,k,l)$, $1 \leq k \leq n/2$, then G contains a k-matching unless $l \leq k-1$, $e(G) = F(n,k,l)$ and either $G = G(n,k,l)$ or $G = G(n,k,k-1)$.*

In the proof of Theorem 3 we shall need a lemma proved in [5] by Erdös and Pósa.

Lemma 3.1 (Erdös and Pósa) *Every graph G of order at least $2k$ such that $\delta(G) \geq k$ contains kK_2.*

The reader may easily check the properties of the functions f and F we give below.

Lemma 3.2 *(i) For* $0 \leq l \leq k-1, \quad 0 \leq k \leq n-1$

$$F(n,k,l) = \begin{cases} f(n,k,l) & \textit{when} \quad 0 \leq l \leq \frac{5k-2n-2}{3} \\ f(n,k,k-1) & \textit{when} \quad \frac{5k-2n-2}{3} \leq l \leq k-1 \end{cases}$$

(ii) $f(n,k,l) - f(n-2,k-1,l-1) = n + l - 2$

(iii) $f(n,k,l) - f(n-2,k-1,l-2) = 2n - 4k + 4l - 2$

Proof of Theorem 3. Our proof starts with the observation that the theorem holds for $k = 1$. Moreover, by the theorem of Erdös and Gallai [4] mentioned in the previous section, it is also true for $l = 0$ and every k.
The proof is by induction on k. Let $k > 1$ and assume that the theorem holds for $k-1$; we shall prove that then it follows also for k. Observe that since $F(n,k,l)$ is weakly decreasing with respect to l when $0 \leq l \leq k-1$, and by Lemma 3.1, we may assume that $\delta(G) = l$. Let u be a vertex of degree $l = \delta(G)$.
Let us assume first that u is adjacent to a vertex v with $d(v,G) = l$. Then $\delta(G - \{u,v\}) \geq l-2$ and $e(G - \{u,v\}) \geq F(n,k,l) - (2l-1)$. There are three possible cases.

Case 1. $l \leq \frac{5k-2n-2}{3}$.
Then $l - 2 \leq \frac{5(k-1)-2(n-2)-2}{3}$, $F(n,k,l) = f(n,k,l)$ and $F(n-2,k-1,l-2) = f(n-2,k-1,l-2)$. Since $l \geq 1$ and $n \geq 2k$ and the function $F(n,k,l)$ is weakly decreasing with respect to l, we apply *(ii)* of Lemma 3.2 to obtain $e(G - \{u,v\}) \geq f(n,k,l) - (2l-1) > f(n-2,k-1,l-2) = F(n-2,k-1,l-2)$.

Case 2. $\frac{5k-2n-2}{3} < l \leq \frac{5k-2n+3}{3}$.

Then, by the similar arguments as above $F(n,k,l) = f(n,k,k-1) > f(n,k,l) \geq f(n-2,k-1,l-2) + 2l - 1 = F(n-2,k-1,l-2) + 2l - 1$.

Case 3. $l > \frac{5k-2n+3}{3}$.

Then, by *(i)* of Lemma 3.2 $F(n,k,l) = f(n,k,k-1) = f(n-2,k-1,k-2) + n - k - 2 = F(n-2,k-1,l-2) + n - k + 3$.

In each case $e(G - \{u,v\}) > F(n,k,l)$ and therefore $G - \{u,v\}$ contains $(k-1)K_2$ by the induction hypothesis. Thus G contains kK_2.

We may now turn to the case when the vertex u is not adjacent to any other vertex of the degree l. Let v be a neighbour of u. We have $\delta(G - \{u,v\}) \geq l - 1$ and $e(G - \{u,v\}) \geq F(n,k,l) - (n + l - 2)$. Also this time we shall have three cases to consider.

Case 1'. $l \leq \frac{5k-2n-2}{3}$.

Then $F(n,k,l) = f(n,k,l) = f(n-2,k-1,l-1)+n+l-2 = F(n-2,k-1,l-1)+n+l-2$. Hence $e(G-\{u,v\}) \geq F(n-2,k-1,l-1)$ and $e(G-\{u,v\}) = F(n-2,k-1,l-1)$ if, and only if, $e(G) = F(n,k,l)$ and $d(v,G) = n-1$. Therefore G contains kK_2 unless $G - \{u,v\} = G(n-2,k-1,l-1)$ and $G = G(n,k,l)$.

Case 2'. $\frac{5k-2n-2}{3} < l \leq \frac{5k-2n}{3}$

Then $F(n,k,l)-(n+l-2) = f(n,k,k-1)-(n+l-2) > f(n,k,l)+(n+l-2)$ and by *(ii)* of Lemma 3.2 we obtain $e(G - \{u,v\}) > f(n-2,k-1,l-1) = F(n-,k-1,l-1)$.

By induction hypothesis $(k-1)K_2 \subset G - \{u,v\}$.

Case 3'. $l > \frac{5k-2n}{3}$

Then $F(n,k,l) = f(n,k,k-1)$. We substitute l by $k-1$ in the formula *(ii)* of Lemma 3.2 to get $f(n,k,k-1) = f(n-2,k-1,k-2)+n-k+3 = F(n-2,k-1,l-1)$.

The rest of the proof runs as in Case 1'.$\square$

It should be noted that Lemma 3.1 has been essentially improved by Brandt in [1].

Theorem 4 (Brandt) *Every graph G of order n with $\delta(G) \geq k$ contains every forest $\mathcal{F}$ of size k and order at most n.*

All results mentioned above suggest that the following conjecture may be true.

Conjecture 1 *If G is a graph of order n with $\delta(G) \geq l$ and $e(G) \geq F(n,k,l)$, then G contains every forest $\mathcal{F}$ of order at most n and size k, unless $l \leq k-1$, $e(G) = F(n,k,l)$, $\mathcal{F} = kK_2$ and either $G = G(n,k,l)$ or $G = G(n,k,k-1)$.*

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