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On self-complementary supergraphs of (n, n)-graphs

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On self-complementary supergraphs of (n, n)-graphs*

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Abstract

We prove that, with one exception, each (n, n)-graph G that is embeddable in its complement has a self-complementary supergraph of order n.

Keywords: graph, embedding, self-complementary graph.

Résumé

Nous démontrons que, avec une seule exception, chaque (n, n)-graphe contenu dans son complément, est sous-graphe d'un graphe auto-complémenté.

Mots-clé: graphe, sous-graphe, graphe auto-complémenté.

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[†]This work was carried out while the first author was visiting University of Orleans

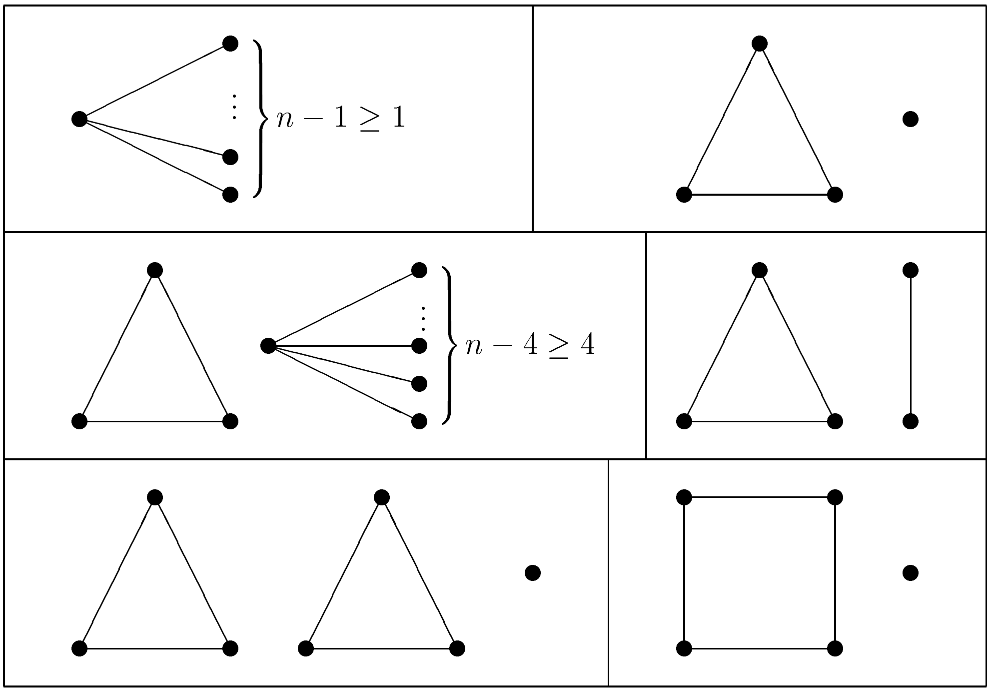


Figure 1: Non-embeddable $(n, n-1)$ graphs.

1 Embeddings of graphs

We shall use standard graph theory notation. A finite, undirected graph G consists of a vertex set $V(G)$ and an edge set $E(G)$.

An embedding of a graph G (into its copmlement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. If there exists an embedding of G we say that G is *embeddable* or that there is *a packing* of 2 copies of the graph G into complete graph K_n.

The following theorem was proved, independently, in [2], [4] and [13].

Theorem 1 *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-2$ then G can be embedded in its complement.* ∎

This result has been improved in many ways. The main references of the paper and of other packing problems are the last chapter of Bollobás's book [2], the 4th Chapter of Yap's book [19] and the survey papers [20] and [17].

The example of the star S_n shows that Theorem 1 cannot be improved by raising the size of G even in the case when G is a tree. However in this case we have the following theorem, proved in [5], which completely characterizes those graphs with n vertices and $n-1$ edges that are embeddable.

Theorem 2 *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-1$ then either G is embeddable or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.* ∎

Remark. Note that the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$ are embeddable.

In this paper we shall consider the case where G is a graph on n vertices with n edges. The more general result known on embeddings of (n, n)-graphs is the following theorem proved in [7].

Theorem 3 *Let $G = (V, E)$ be a graph of order n. If $|E(G)| = n$ then either G is embeddable or G is isomorphic to one of the graphs of Figure 2.* ■

We are interested in these improvements of above mentioned theorems that deal with some additional information about embeddings *i.e.* packing permutations.

We shall need some additional terminology in order to formulate the result.

2 Self-complementary permutations

A graph G is *self-complementary* (briefly, s-c) if it is isomorphic to its complement (cf. [11], [12], or [10]). It is clear that an s-c graph has $n \equiv 0, 1 \pmod 4$ edges. We extend the above definition to the case where $n \equiv 2, 3 \pmod 4$ as follows. A graph G of order $n \equiv 2, 3 \pmod 4$ is *almost self-complementary* (or briefly, a-s-c) if G is of size $\frac{1}{2}(\binom{n}{2} - 1)$ and G is a subgraph of its complement.

We also say that there exists a packing of G, G, K_2 (into K_n) or that G is self-complementary in $K_n - e$ (cf. [6])

In this paper we shall use the term 'self-complementary' also in the case $n \equiv 2, 3 \pmod 4$.

The following theorem characterizes the structure of *self-complementary permutation* that is a permutation which transforms one copy of a self-complementary graph into another. The part concerning the cases $n \equiv 0, 1 \pmod 4$ was proved in [11] and [12]. The part concerning the cases $n \equiv 2, 3 \pmod 4$ was proved in [6].

Theorem 4 *Let $G = (V, E)$ be a self-complementary graph of order n, and let σ be a self-complementary permutation of G. Then*

when $n \equiv 0 \pmod 4$, *σ consists of cycles of lengths that are multiples of 4.*

when $n \equiv 1 \pmod 4$, *σ consists of cycles of lengths that are multiples of 4, except for one cycle of length one.*

when $n \equiv 2 \pmod 4$, *then either*

- *σ has two fixed points and the other cycles have lengths that are multiples of 4, or*

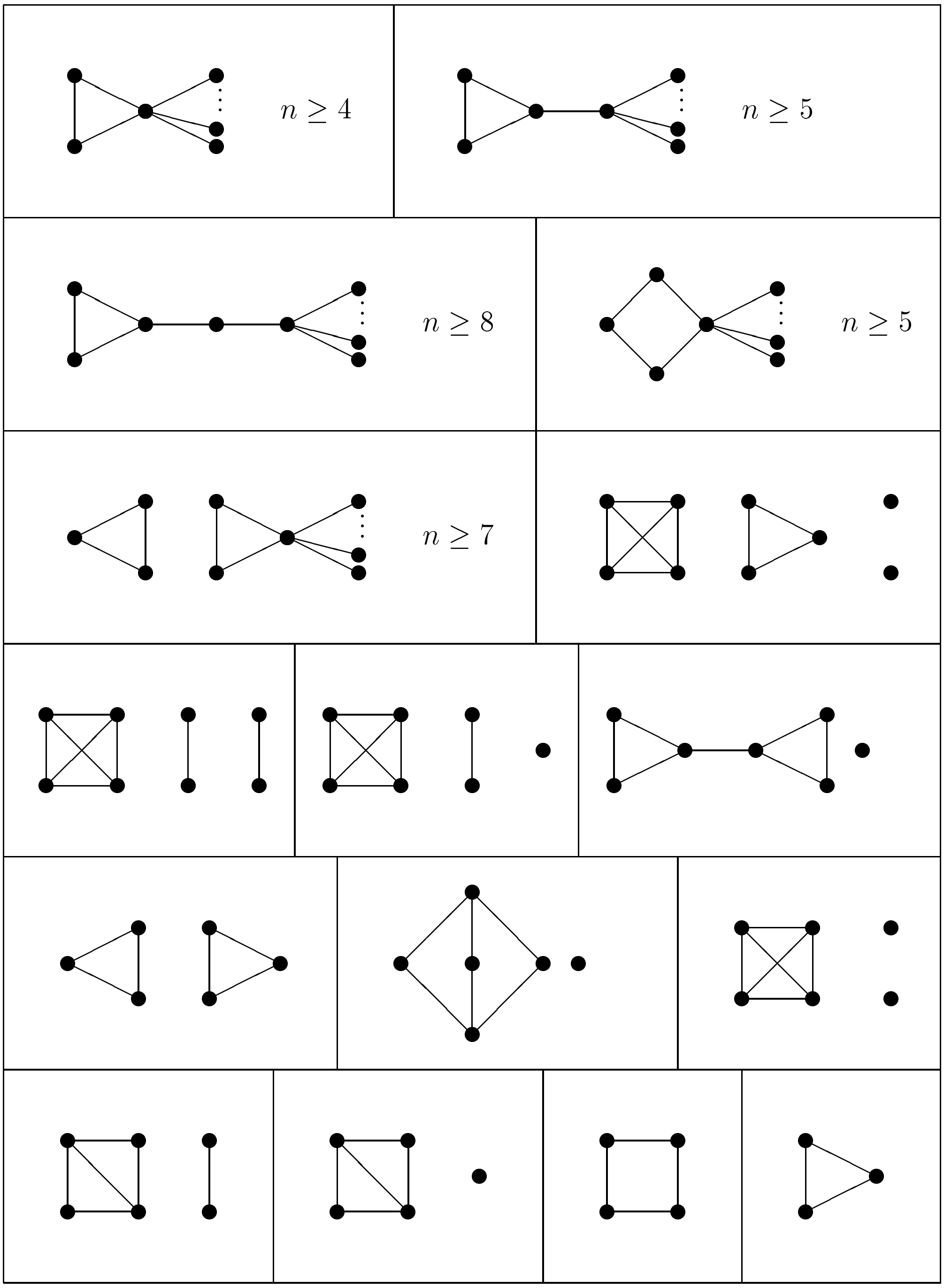


Figure 2: Non-embeddable (n, n) graphs.

- *σ consists of a cycle of length $4h+2$ and the other cycles have lengths that are multiples of 4.*

when $n \equiv 3 \pmod 4$, *then either*

- *σ consists of a cycle of length 3 and the other cycles have lengths that are multiples of 4, or*
- *σ consists of a cycle of length $4h+2$ and the other cycles have lengths that are multiples of 4, and one fixed point.*

■

It is easy to see that there exists a packing of two graphs G_1 and G_2 if and only if G_1 is a subgraph of $\overline{G_2}$ (or, by symmetry, G_2 is a subgraph of $\overline{G}_1$). It is evident that subgraphs of self-complementery graphs are embeddable. It is known that for (n,k)-graphs with $k \leq n-1$ the property 'be embeddable' and the property 'be a subgraph of a self-complementery graph' are in fact equivalent. More precisely we have

Theorem 5 *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-1$ then either there exists an embedding σ of G having all its cycles of length 4, except for*
one cycle of length one if $n \equiv 1 \pmod 4$,
one cycle of length two or two cycles of length one if $n \equiv 2 \pmod 4$,
two cycles of lengths one and two if $n \equiv 3 \pmod 4$,
or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3, n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.

The part concerning the cases $n \equiv 0, 1 \pmod 4$ was proved in [1]. The existence of an almost-self-complementary packing (i.e. the cases $n \equiv 2, 3 \pmod 4$) was proved in [15]. In both cases the proofs use the cyclic structure of the packing permutation and are based on the notion of a *good permutation* defined bellow. Let G be a graph of order n. Throughout this paper we shall use the following terminology. A permutation σ of $V(G)$ will be called good for G if σ is an embedding of G and all its cycles have length 4, except for one cycle of length one if $n \equiv 1 \pmod 4$, one cycle of length two or two cycles of length one if $n \equiv 2 \pmod 4$, two cycles of lengths one and two if $n \equiv 3 \pmod 4$.

The good permutations are useful because if we are able to prove that the packing permutation σ of an embeddable graph G is good, then the following procedure allows us to construct a self-complementary supergraph H of G: We shall construct two graphs H and H' with $V(H) = V(H') = V(G)$:

(i) For every edge $e \in E(G)$, assign the edges e and $\sigma^2(e)$ to $E(H)$ and assign the edges $\sigma(e)$ and $\sigma^3(e)$ to $E(H')$.

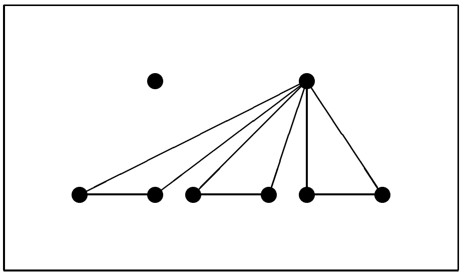


Figure 3: An emdeddable graph without s-c permutaion.

(ii) If after step (i) there exist two vertices x, y of G such that the edge $f = xy$ does not belong to $E(H) \cup E(H')$, then assign it as well as the edge $\sigma^2(f)$ to $E(H)$ and the edges $\sigma(f)$ and $\sigma^3(f)$ to $E(H')$.

(iii) Repeat step (ii) until all edges of the complete graph constructed on $V(G)$ lie in $E(H) \cup E(H')$.

Clearly, the graph H defined by the above method is an s-c graph and the graph H' is isomorphic to H. Moreover, σ is an s-c permutation which maps H onto H'. Finally, we clearly have $G \subset H$.

Remark. Observe that the above procedure cannot be used in general, if σ is just a packing permutation.

Using this terminology, Theorem 5 says that if $|E(G)| \leq n - 1$ and G is embeddable then there exists a good permutation for G and, in consequence, G is subgraph of a self-complementary graph. The aim of this paper is to study the case $|E(G)| = n$.

3 Result

As observed above, it is evident that subgraphs of self-complementery graphs are embeddable. In general the converse is not true. For, consider the graph on $2k + 2$, vertices with vertex set, $V = \{u, v, x_1, \ldots, x_k, y_1, \ldots, y_k\}$, k odd, $k \geq 3$, and edge set $E = \{vx_i, vy_i, x_iy_i\}$, $i = 1, 2, \ldots, k$.

Thus, the vertex u is an isolated vertex, the vertex v is of degree $2k$ while the remaining vertices are of degree two. The permutation $(uv)(x_1x_2 \ldots x_k)$ provides an embedding permutation which is however not an s-c permutation because of the transposition (uv). On the other hand, in each embedding permutation the vertex v have to be mapped on u and none of vertices of degree two can be put onto v. Therefore, each embedding permutation contains a transposition in its cycle structure. The smallest graph of this family is given in Fig. 3. Observe that this is an $(8, 9)-$graph.

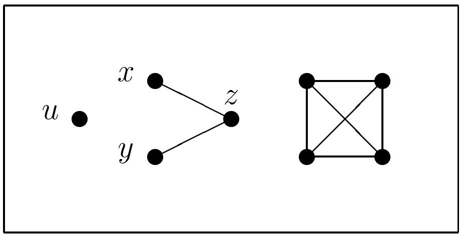


Figure 4: The exceptional graph F_0.

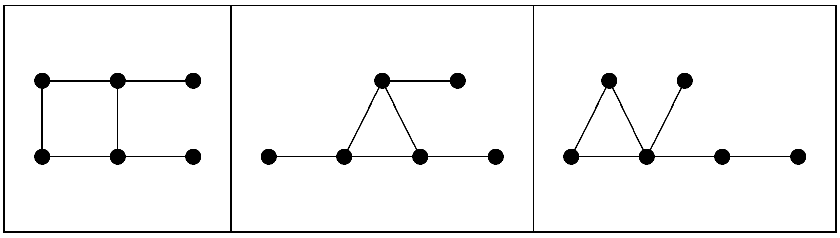


Figure 5: The special graphs.

There is also an $(n, n)-$graphs which is embeddable but is not a subgraph of any s-c graph. For, consider the graph F_0 drawn in Fig. 4. Let σ be an embedding of F_0. It is easy to see that the set of the images of the vertices $\{a, b, c, d\}$ of K_4 must contain the vertices u, x, y and one of the vertices of K_4. In this situation, the vertex z have to be mapped on itself *i.e.* σ must have a fixed point, and therefore cannot be an s-c permutation.

The aim of this paper is to show that the graph F_0 is the unique (n, n)-graph with this property. The main result is the following theorem.

Theorem 6 *Every embeddable graph of order n and size at most n is a subgraph of an s-c graph of order n except for the graph F_0 defined in Fig. 4.*

More precisely, for some technical reasons, we shall prove the following stronger result.

Theorem 7 *Let G be a graph of order n and size at most n. If G is embeddable, then either there exists a good permutation for G, or G is isomorphic to the graph F_0 defined in Fig. 4, or else $n = 6$ and G is isomorphic to one of the special graphs defined in Fig. 5.*

It is easy to see that every special graph is a subgraph of the unique s-c graph on six vertices having only cyclic s-c permutation.

The rest of the paper is organized as follows: in the next section we state a lemma. The proof of Theorem 7 is by induction with respect to the order n. Sections 5 and 6 contain small cases *i.e.* $n = 5, 6, 7$ and $n = 8$. The main part of the proof is provided is Section 7.

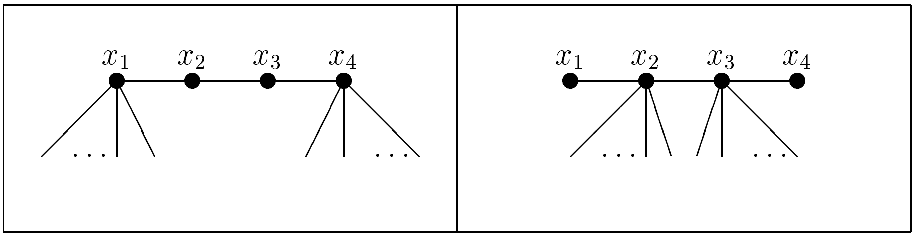


Figure 6: Two layouts

4 A lemma

The main tool in the proof will be the following lemma.

Lemma 8 *Given four vertices x_i, $i = 1,...,4$ in G, such that at least one of the following conditions holds:*
A) $N(x_i) \subset \{x_{i-1}, x_{i+1}\}$ for $i = 2,3$ and $x_1x_4 \notin E(G)$,
B) $N(x_1) \subset \{x_2\}$, $N(x_4) \subset \{x_3\}$ (see Fig. 6).
If $G - \{x_i : i = 1, ..., 4\}$ has a good permutation σ' then $(x_1x_2x_4x_3)\sigma'$ is a good permutation of G. Moreover, if $x_1,...,x_4$ cover at least 6 edges then G has a good permutation. If $x_1,...,x_4$ cover 5 edges then G' has a good permutation unless G' is not an embeddable graph. ■

An easy proof of Lemma 8 is left to the reader.

5 Case $n \leq 7$

The proof of Theorem 7 is by induction on n. The case $n \leq 5$ is easy and is left to the reader. For $n = 6$ or $n = 7$ the theorem can be verified directly as it is done below. In Fig. 7 we give s-c graphs of order 6, while in Fig. 8, 9 and 10 all graphs of order 6 are represented with the references to the corresponding s-c graphs, unless a given graph is non-embaddable.
For $n = 7$ the corresponding figures are Fig. 11 and Fig. 12 – 15.

6 Case $n = 8$

Let G be embeddable, non exceptional and non special graph of order 8. Without loss of generality we may assume that $|E(G)| = 8$. For $W \subset V(G)$ by $G - W$ we shall denote the subgraph of G induced by $V(G) - W$. We shall distinguish several cases where we shall find a subset W of $V(G)$ such that the graph $G - W$ has a good permutation σ' and then we shall extend σ' to a good permutation σ of G.

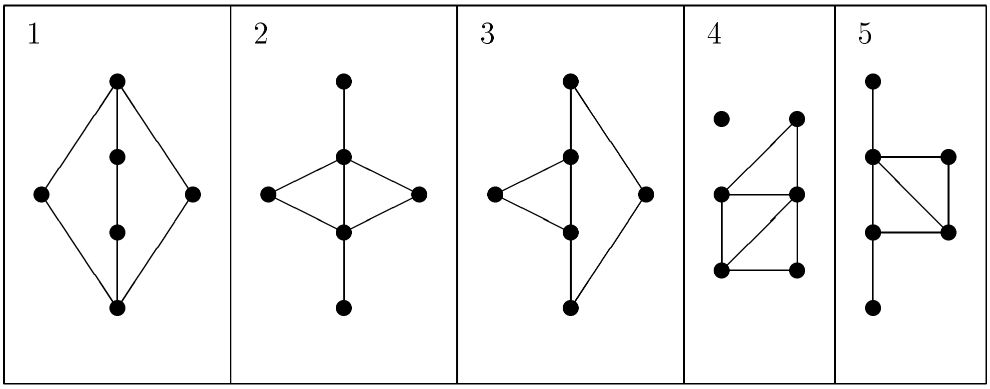


Figure 7: Self-complementary graphs of order 6.

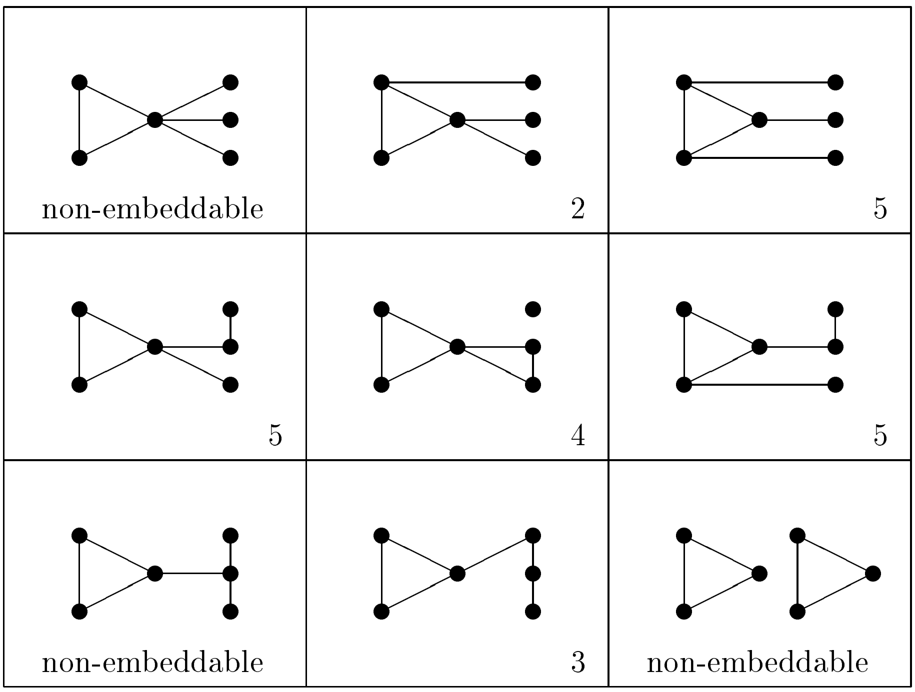


Figure 8: (6,6)-graphs with cicumference 3.

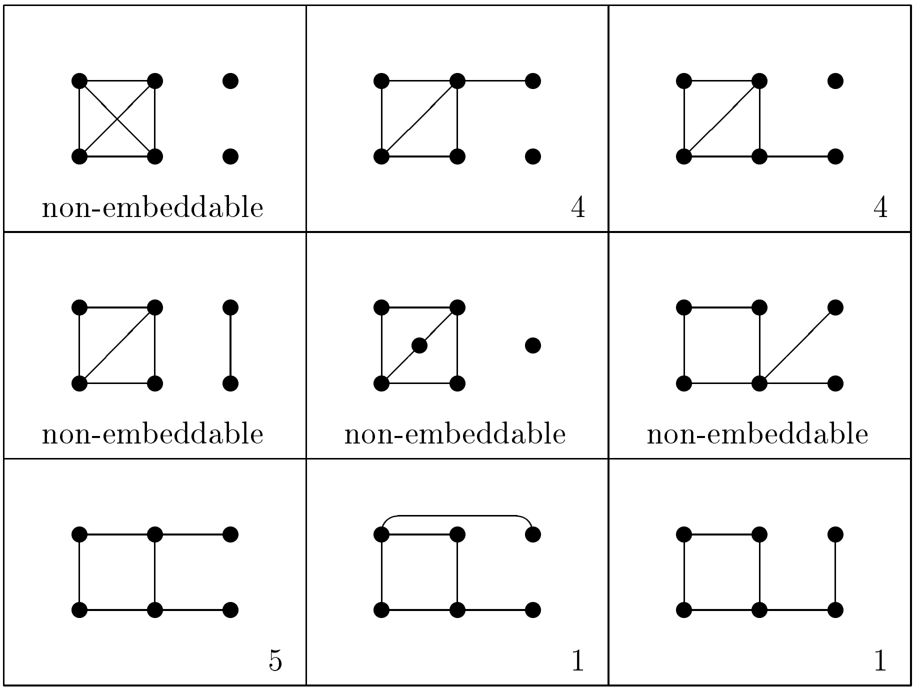


Figure 9: (6,6)-graphs with cicumference 4.

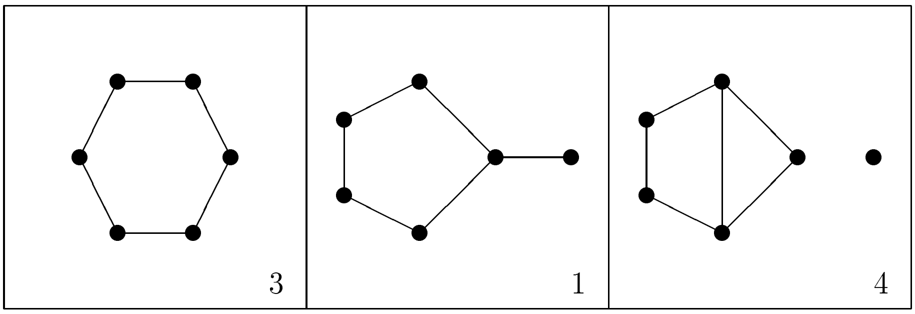


Figure 10: (6,6)-graphs with circumference 6 and 5.

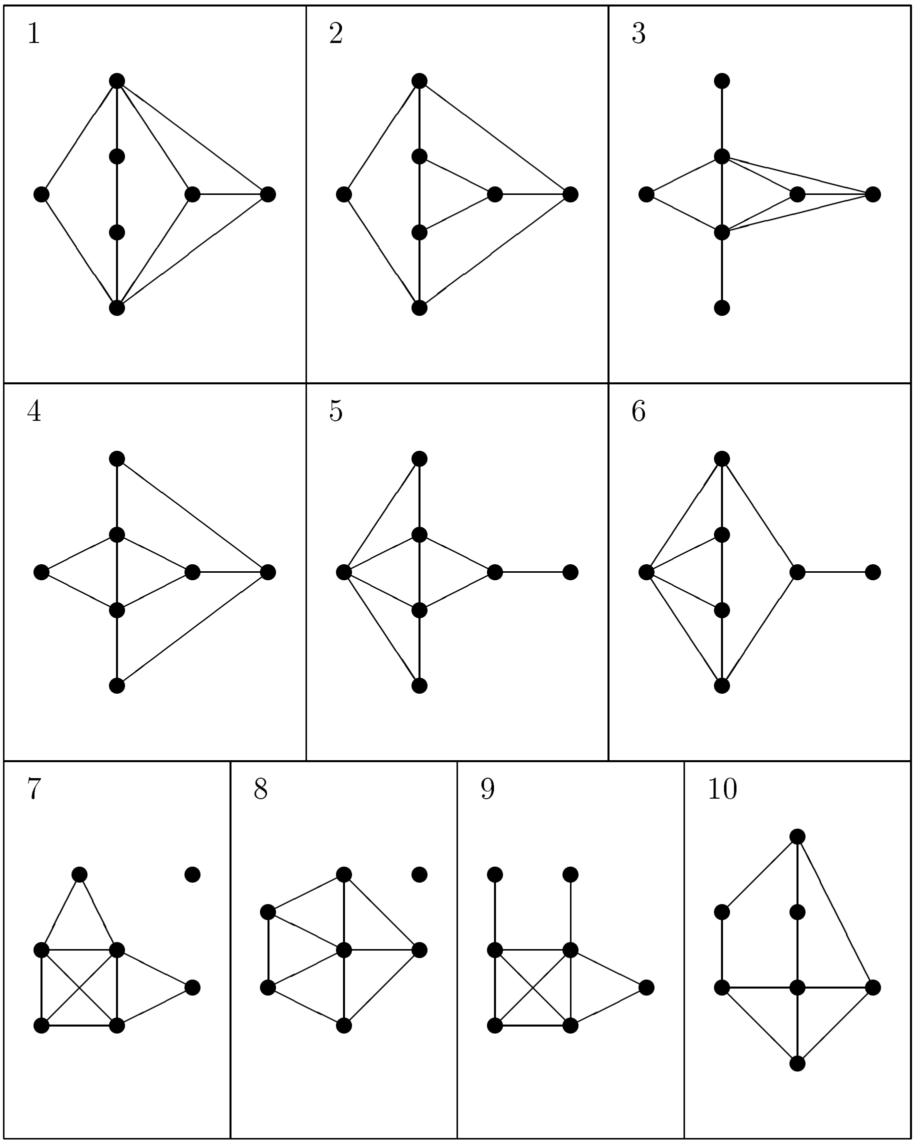


Figure 11: Self-complementary graphs of order 7.

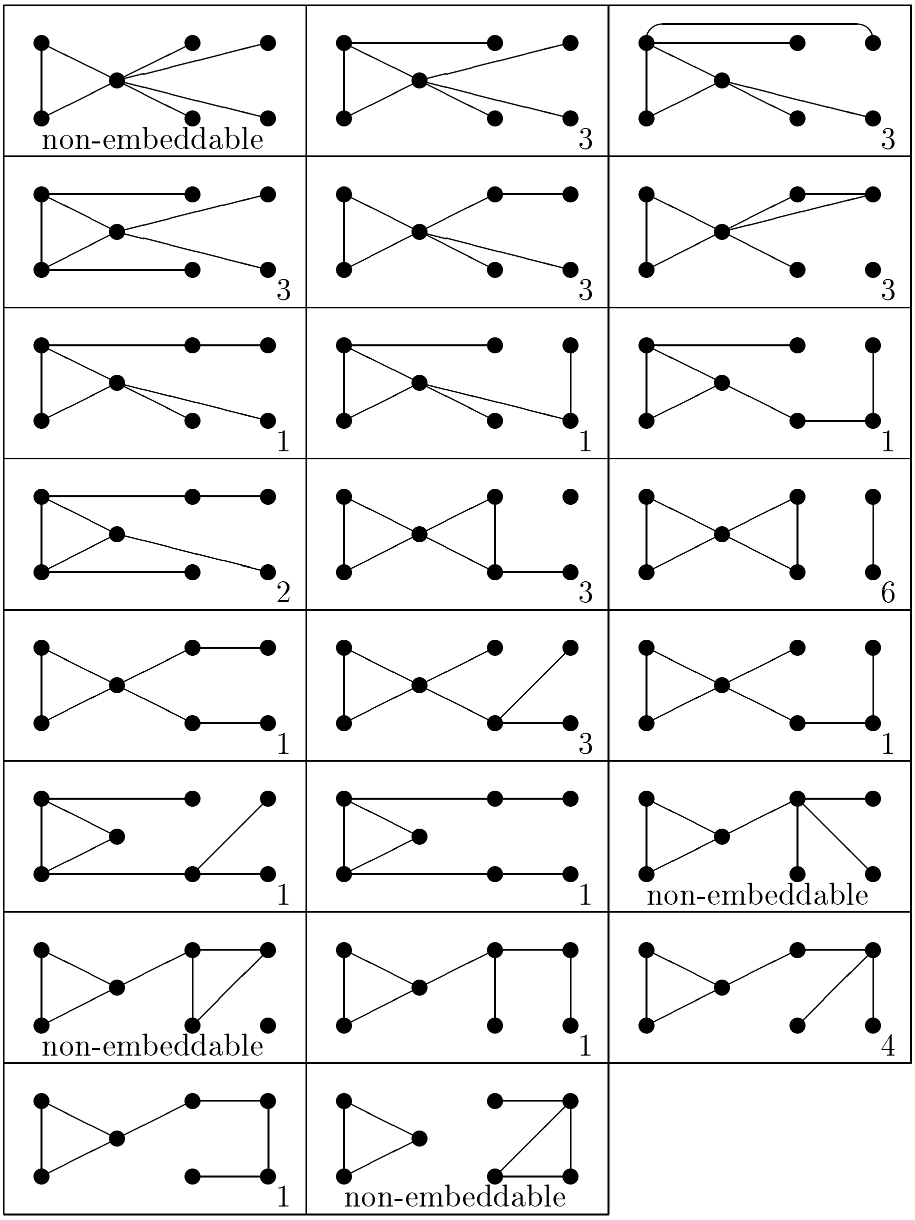


Figure 12: (7,7)-graphs with cicumference 3.

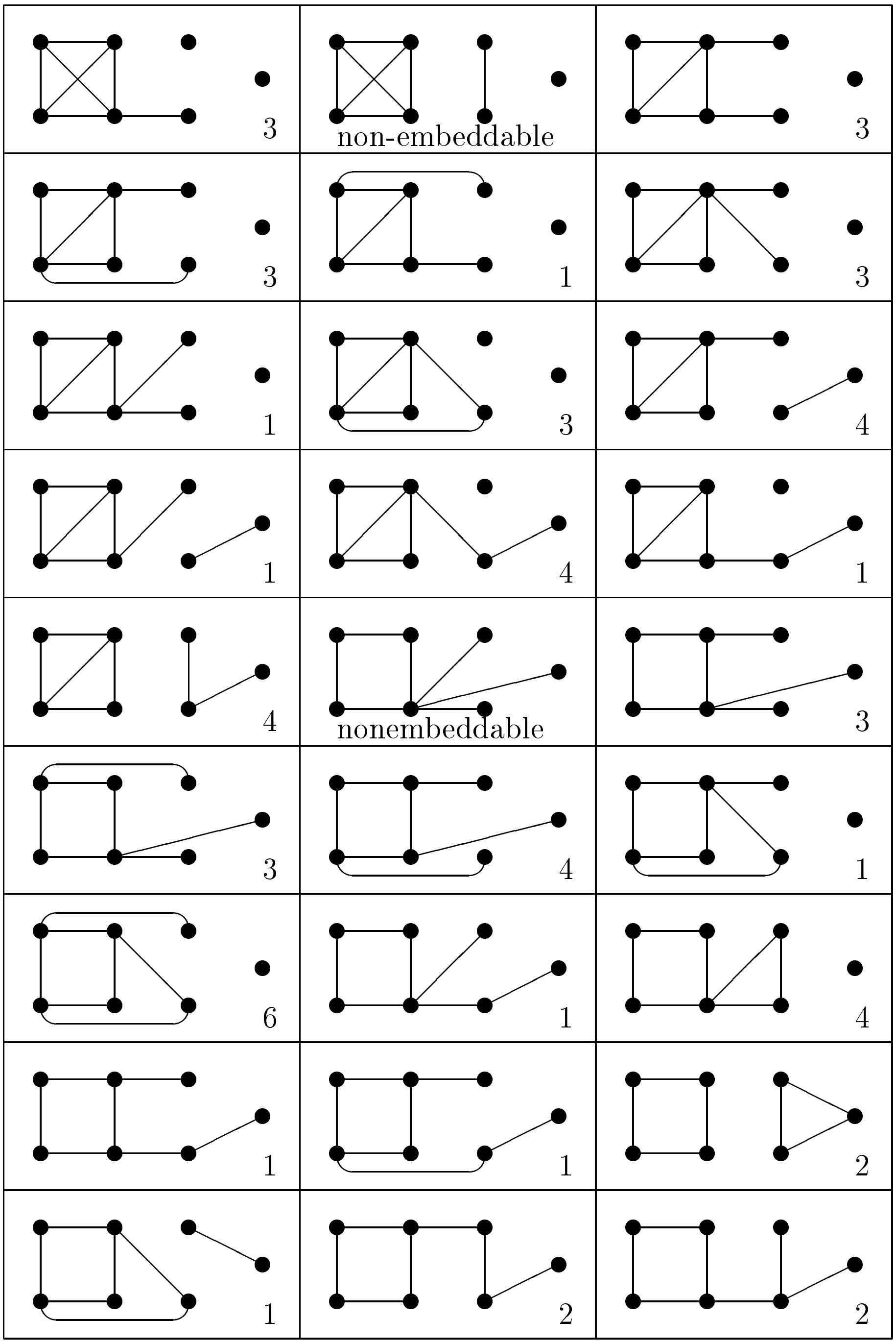


Figure 13: (7,7)-graphs with cicumference 4.

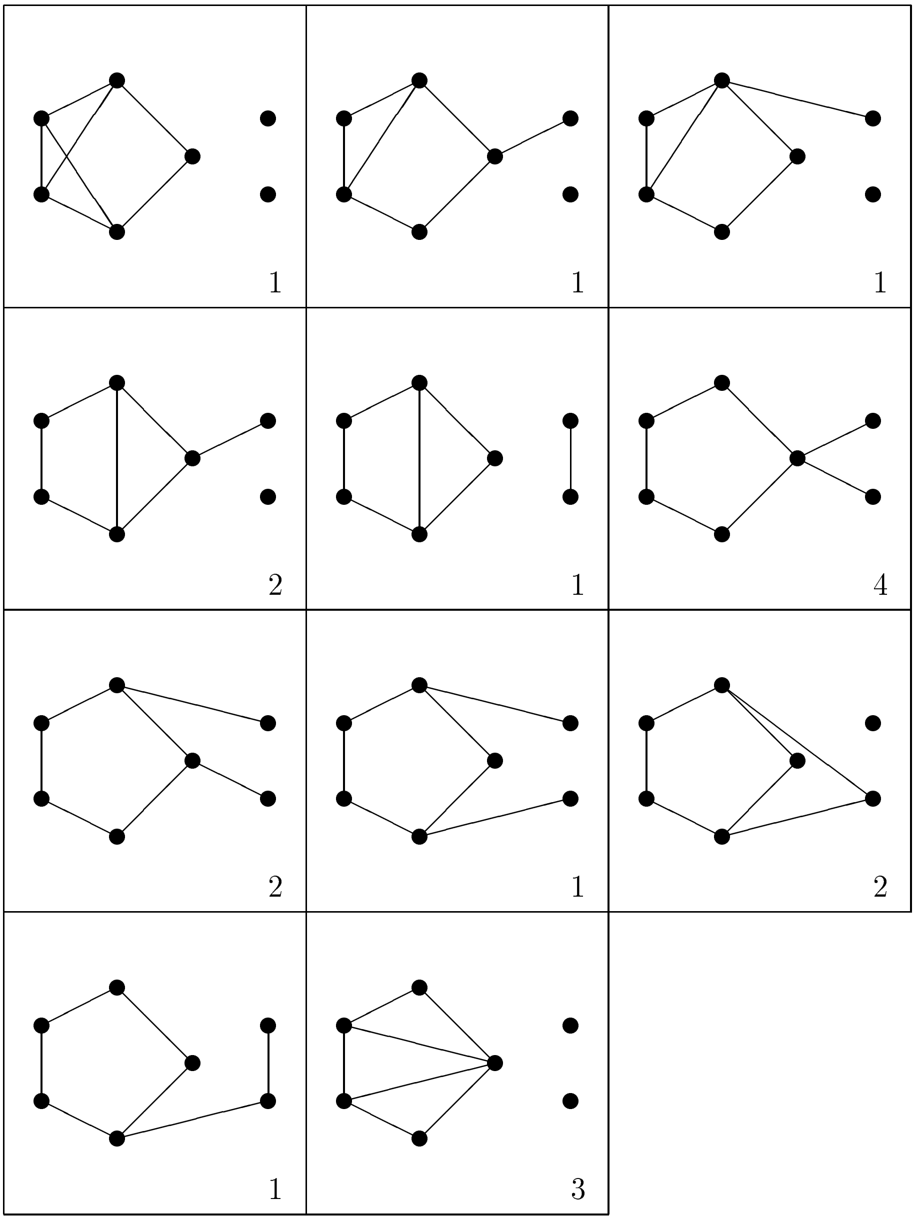


Figure 14: (7,7)-graphs with circumference 5.

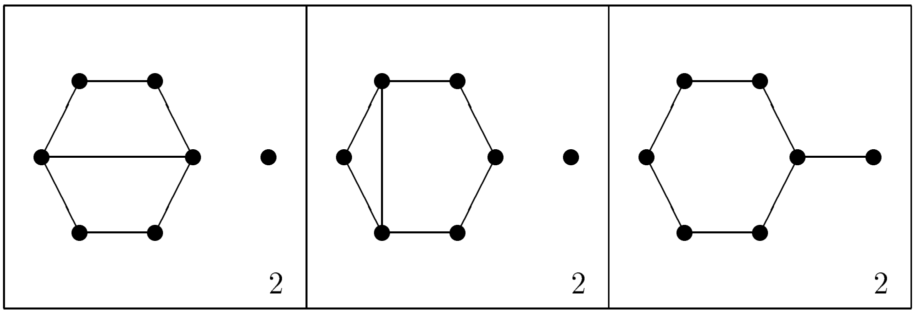


Figure 15: (7,7)-graphs with circumference 6.

Observe that in the case where vertices x_1,...,x_4 satisfy condition A or B of Lemma 8 and cover 5 edges then G has a good permutation unless $G' = G - \{x_i : i = 1, ..., 4\}$ is a star or the union of an isolated vertex and a triangle.

I. There are at least two isolated vertices u_1, u_2 in G.

For every $x_i \in V(G) - \{u_1, u_2\}$ write $d_i = d(x_i)$, $i = 1$,...,6. Without loss of generality we may assume that $d_1 \geq d_2 \geq ... \geq d_6$. It is clear that $16 = \sum_{i=1}^{6} d_i \leq 3(d_1 + d_2)$, hence that $d_1 + d_2 \geq 6$. If $d_1 + d_2 > 6$ then the vertices u_1, x_1, x_2, u_2 cover at least 6 edges and satisfy condition A of Lemma 8, and then G has a good permutation. If $d_1 + d_2 = 6$ then $6 \geq d_3 + d_4 \geq d_5 + d_6$ and $d_3 + d_4 + d_5 + d_6 = 12$, $d_3 + d_4 = d_5 + d_6 = 6$. Hence $d_i = 3$ for $i = 1$,...,6. Therefore there are at least two nonadjacent vertices of degree 3 in G. Since these two vertices with u_1, u_2 satisfies condition A of Lemma 8 and cover 6 edges, G has a good permutation.

II. There is exactly one isolated vertex u and there is at least one vertex v of degree 1 in G.

Let s be the only neighbour of v. Observe that for any vertex $x \in V(G) - \{u, v, s\}$, the vertices v, s, x, u satisfy condition B of Lemma 8. Therefore it is enough to consider the case when for any vertex $x \in V(G) - \{u, v, s\}$, the vertices s, x cover at most 5 edges *i. e.* any vertex nonadjacent to s has its degree at most $5 - d$ and any vertex adjacent to s has a degree at most $6 - d$, where $d = d(s)$. Calculating the sum of the degrees of the vertices of G we obtain that $16 \leq 0 + 1 + d + (d - 1)(6 - d) + (6 - d)(5 - d)$. Hence $d(s) = d \leq 3$.

Let us suppose that $d(s) = 3$. It is easily seen that every neighbour of s, except for v, is of degree 3, and the vertices nonadjacent to s have a degree 2. Let $x \in V(G) - \{u, v, s\}$ be a neighbour of s. the vertices s and x cover 5 edges and $G' = G - \{u, v, s, x\}$ is not a star and is not the union of an isolated vertex and a triangle.

Let us suppose that $d(s) = 2$. Then the second neighbour of s has the

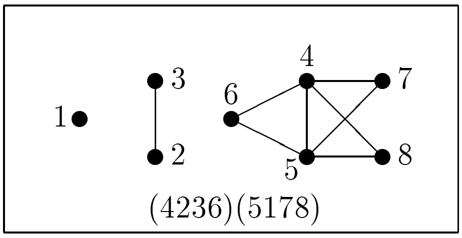


Figure 16: An (8,8)-graph with a good permutation.

degree at most 4 and four vertices nonadjacent to s are of degree at most 3. Let $x \in V(G) - \{u, v, s\}$ be the second neighbour of s and let $y_i \in V(G) - \{u, v, s\}$, $i = 1, ..., 4$ be vertices nonadjacent to s. Observe that for y_i, $i \in \{1, ..., 4\}$ nonadjacent to x, the vertices y_i, v, s, x satisfy condition A of Lemma 8. Since u is the only isolated vertex, if $d(x) = 4$ and y_1 is the vertex nonadjacent to x then the vertices y_1, v, s, x cover at most 6 edges. It is easy to check that since $d(y_i) \leq 3$, $i = 1, ..., 4$, if $d(x) = 3$ or $d(x) = 2$ then there is the vertex y_i, $i \in \{1, ..., 4\}$ nonadjacent to x such that y_i, v, s, x cover at least 6 edges. If $d(x) = 1$ then $d(y_i) = 3$ for $i = 1, ..., 4$ and we obtain the exceptional graph.

Finally, let us suppose that $d(s) = 1$. Without loss of generality we may assume that every vertex of degree 1 is a vertex of an isolated edge. Thus every vertex from $V(G) - \{u, v, s\}$ has a degree at least 2. Observe that $4 \geq \Delta(G) \geq 3$. Let us consider the case that there is a vertex $x_1 \in V(G) - \{u, v, s\}$ such that $d(x_1) = 4$. The vertices u, v, s, x_1 satisfy condition A of Lemma 8 and cover 5 edges. It is easy to see that $G - \{u, v, s, x_1\}$ is not embeddable if G is isomorphic to the graph given in Fig. 16. Let us suppose that $\Delta(G) = 3$. Let $x_1 \in V(G) - \{u, v, s\}$ such that $d(x_1) = 3$. Let x_2 be the only vertex from $V(G) - \{u, v, s\}$ nonadjacent to x_1. Since $d(x_2) \geq 2$, the vertices x_1, v, s, x_2 satisfy condition A of Lemma 8 and cover at least 6 edges.

III. There is exactly one isolated vertex u and there are no vertices of degree 1 in G.

Calculating degrees of vertices we obtain that either $\Delta(G) = 3$ and there are two vertices of degree 3 and five vertices of degree 2 or $\Delta(G) = 4$ and there is one vertex of degree 4 and six vertices of degree 2. If there are two vertices x_1, x_2 of degree 3 then either x_1, x_2 are joined by a path (including the edge x_1x_2) and x_1 is the vertex of a cycle different from a cycle with the vertex x_2 or x_1, x_2 are joint by three different paths. If there is a vertex y_1 of degree 4 then y_1 is the only common vertex of two different cycles. It is easy to check that either there are four vertices satisfy condition A of Lemma 8 and cover at least 6 edges or G is isomorphic to one of graphs given in Fig. 17.

IV. There are no isolated vertices and there are two vertices v_1, $v_2 \in$

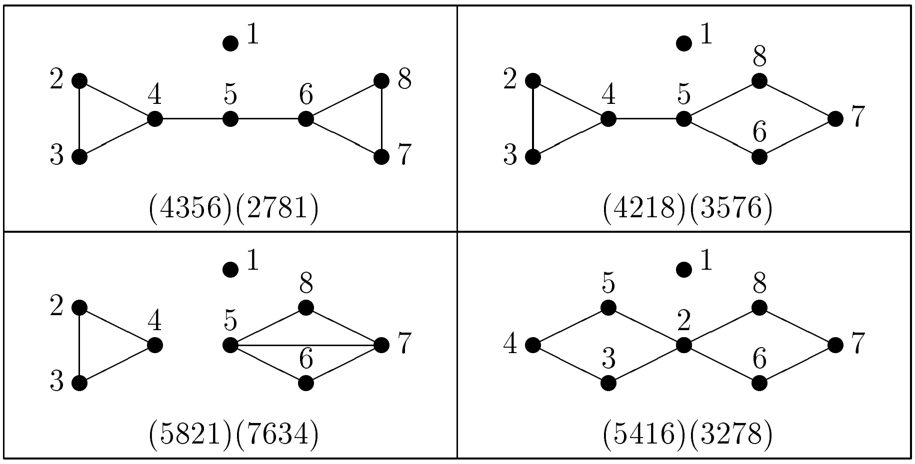


Figure 17: Four (8,8)-graphs with good permutations.

$V(G)$ **such that** v_1v_2 **is an isolated edge.**

The remaining six vertices cover 7 edges. Thus it is clear that $3 \leq \Delta(G) \leq 5$. Let $x_1 \in V(G)$ such that $d(x_1) = \Delta(G)$. If x_1 is the neighbour of every vertex from $V(G) - \{v_1, v_2\}$ then there is a vertex $x_2 \in V(G) - \{v_1, v_2\}$ such that $d(x_2) = 1$ and hence the vertices v_1, v_2, x_1, x_2 satisfy condition B of Lemma 8 and cover 6 edges. If $d(x_1) = 4$ then let x_3 be the only vertex of $V(G) - \{v_1, v_2\}$ nonadjacent to x_1. Hence the vertices x_1, v_1, v_2, x_3 satisfy condition A of Lemma 8 and cover at least 6 edges. Finally, let us suppose that $d(x_1) = 3$. Let $x_2, x_3 \in V(G) - \{v_1, v_2\}$ be vertices nonadjacent to x_1. Since the vertices x_1, v_1, v_2, x_2 and the vertices x_1, v_1, v_2, x_3 satisfy condition A of Lemma 8, it is enough to consider the case when for $j = 2, 3$, the vertices x_1, v_1, v_2, x_j cover 5 edges, *i. e.* $d(x_j) = 1$, and $G - \{x_1, v_1, v_2, x_j\}$ is not embeddable. Thus G is isomorphic to the union of two isolated edges and K_4 and hence G is not embeddable.

V. There are no isolated vertices and isolated edges in G **and there are two vertices** v_1, v_2 **of degree 1 such that** $N(v_1) \neq N(v_2)$**.**

Let y_i be the only neighbour of v_i for $i = 1, 2$. Since the vertices v_1, y_1, y_2, v_2 satisfy condition B of Lemma 8, it is enough to consider the case when the vertices y_1, y_2 cover at most 5 edges. Let c be the number of edges covered by y_1 or y_2. It is clear that $c \geq 3$ and that without loss of generality we may assume that for every pair of vertices v_p, v_q of degree 1 such that $\{y_q\} = N(v_q) \neq N(v_p) = \{y_p\}$, the vertices y_p, y_q cover at most c edges.

Let us suppose that $c = 5$. Then it is enough to consider the case when $G - \{v_i, y_i, i = 1, 2\}$ is a star or the union of an isolated vertex and a triangle. In such a case if $d(y_i) = 3$, $i = 1, 2$ then $y_1y_2 \in E(G)$ and G is a graph isomorphic to one of the graphs in Fig. 18. Let us suppose that $d(y_1) = 2$ and x is the

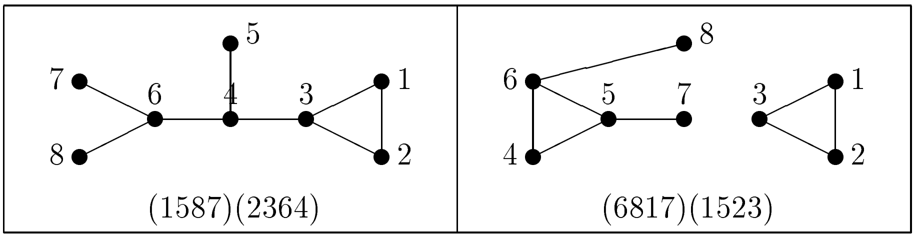


Figure 18: Two (8,8)-graphs with exemplary permutations.

neighbour of y_1, $x \neq v_1$. Thus $x \in V(G) - \{v_1, y_1\}$. It is easy to check that in this case $G - \{v_i, y_i, i = 1, 2\}$ is the union of an isolated vertex and a triangle, and that there is a vertex $y \in V(G) - \{v_1, y_1, x\}$ such that y is not a neighbour of x and $d(x) + d(y) \geq 5$. Therefore the vertices y, v_1, y_1, x satisfy condition A of Lemma 8 and cover at least 6 edges.

Let us suppose that y_1, y_2 cover 4 edges. Without loss of generality we may assume that $d(y_1) \leq d(y_2)$. It is clear that in this case $d(y_1) = 2$. Let x be the neighbour of y_1, $x \neq v_1$. Thus $x \in V(G) - \{v_1, y_1\}$. Since $G - \{v_i, y_i; i = 1, 2\}$ has 4 edges, we can check that there is a vertex $y \in V(G) - \{v_1, y_1, x\}$ such that y is not a neighbour of x and $d(x) + d(y) \geq 5$. Hence the vertices y, v_1, y_1, x satisfy condition A of Lemma 8 and cover at least 6 edges.

Let us suppose that y_1 and y_2 cover 3 edges. In this case $d(y_i) = 2$, $i = 1, 2$ and $y_1 y_2 \in E(G)$. There is a vertex $y \in V(G) - \{v_i, y_i; i = 1, 2\}$ such that $d(y) = 3$. Hence vertices y, v_1, y_1, y_2 satisfy condition A of Lemma 8 and cover 6 edges.

VI. There is no isolated vertex in G, there is at least one vertex of degree 1, and all the vertices of degree 1 have a common neighbour w with $d(w) > 1$.

It is obvious that $\Delta(G) \leq 7$. Set $L_i = \{u \in V(G) : u \neq w, d(u) = i\}$ and $l_i = |L_i|$, $i = 1, ..., 7$. Write $r = d(w) - l_1$. Calculating the sum of the degrees of the vertices of G we obtain $16 = (r + l_1) + \sum_{i=1}^{7}(il_i)$. Since $\sum_{i=1}^{7} l_i = 7$, $2 = r + \sum_{i=3}^{7} l_i(i-2)$ and finally $l_i = 0$ for $i = 5, 6, 7$ and $2 = l_3 + 2l_4 + r$. Hence there are the following possibilities:
a) $l_4 = 1$, $l_3 = r = 0$,
b) $l_4 = 0$, $l_3 = 2$, $r = 0$,
c) $l_4 = 0$, $l_3 = r = 1$,

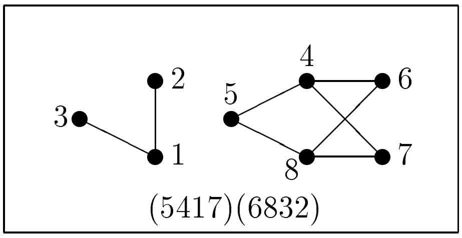


Figure 19: An (8,8)-graph with a good permutation.

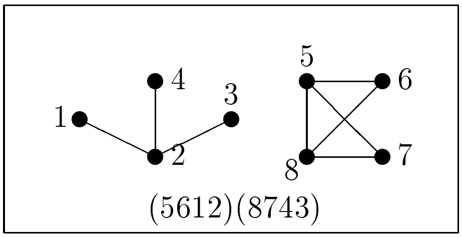


Figure 20: An (8,8)-graph with a good permutation.

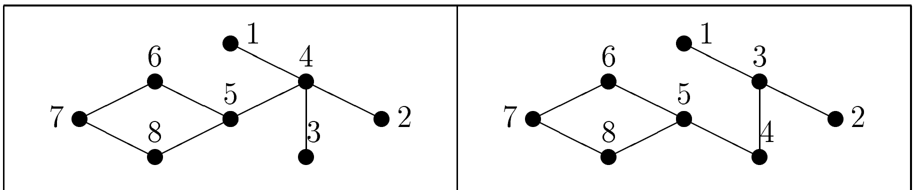
d) $l_4 = l_3 = 0$, $r = 2$.
Since $l_2 = 7 - (l_4 + l_3 + l_1)$ and $d(w) = l_1 + r \geq 2$, we have

- $l_2 = 4$ in case (a),
- $l_2 \leq 3$ in case (b),
- $l_2 \leq 5$ in case (c) and
- $l_2 \leq 6$ in case (d).

Case (a). Observe that there are exactly two vertices v_1, v_2 of degree 1. Let y be the only vertex of degree 4 in G. Since $N(w) = \{v_1, v_2\}$, the vertices y, v_1, w, v_2 satisfy condition A of Lemma 8 and cover 6 edges.

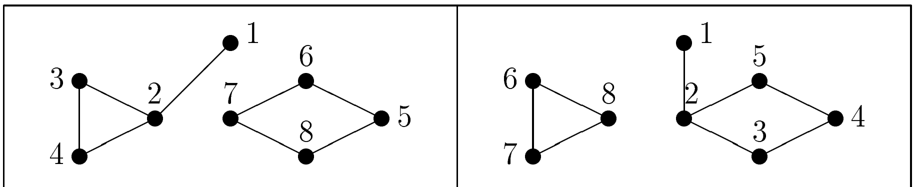
Case (b). Observe that in this case either $d(w) = 2$ or $d(w) = 3$.Let $y \in V(G) - \{w\}$ be a vertex of degree 3. If $d(w) = 2$ then the vertices y, v_1, w, v_2 satisfy condition A of Lemma 8 and cover 5 edges. If $G - \{y, v_1, w, v_2\}$ is not embeddable then G is isomorphic to the graph in Fig. 19. If $d(w) = 3$ then G is isomorphic to the graph in Fig. 20.

Case (c). Let $y \in V(G) - \{w\}$ be a vertex of degree 3. Observe that the vertices w and y are joined by a path P (including the edge wy), y is a vertex of a cycle C contained in G, $l_1 + l_2 = 6$ and $l_1 \geq 1$. If $|V(C)| \geq 5$ then the vertices y, x_1, x_2, x_3 satisfy condition A of Lemma 8 and cover 6 edges, where $x_1, x_2, x_3 \in V(C)$ and $yx_1, x_1x_2, x_2x_3 \in E(C)$. If $|V(P)| \geq 4$ then the vertices y, z_1, z_2, z_3 satisfy condition A of Lemma 8 and cover at least 6 edges, where $z_1, z_2, z_3 \in V(P)$ and $yz_1, z_1z_2, z_2z_3 \in E(P)$. Therefore we can assume that $|V(C)| \leq 4$ and $|V(P)| \leq 3$. If there is another cycle C^1 contained in G, *i. e.* if



(6721)(8534) (6713)(8524)

Figure 21: Two (8,8)-graphs with good permutations.



(5821)(7643) (5476)(3281)

Figure 22: Two (8,8)-graph with exemplary good permutations.

$|V(C)| = 3$, $P = wy$, $l_1 = 1$ and $|V(C^1)| = 3$ then the vertices x_1, v_1, w, y satisfy condition A of Lemma 8 and cover 6 edges, where $d(v_1) = 1$, x_1 is a vertex of C^1. If C is the unique cycle contained in the embeddable graph G then G is isomorphic to one of graphs in Fig. 21.

Case (d). Observe that w is a vertex of a cycle C contained in G, $l_1 + l_2 = 7$, $l_1 \geq 1$. Similarly as in case (c) we may assume that $|V(C)| \leq 4$. Then the embeddable graph G is isomorphic to one of graphs in Fig. 22.

VII. The graph G is a union of vertex disjoint cycles.

Let C_l denote a cycle of order l. Then either $G = C_8$, or $G = C_3 \cup C_5$, or else $G = C_4 \cup C_4$. If $G = C_8$ then the vertices x_i, $i = 1, ..., 4$ ($x_j x_{j+1} \in E(G)$, $j = 1, ..., 3$) satisfy condition A of Lemma 8, cover 5 edges and $G - \{x_i, i = 1, ..., 4\}$ is embeddable. If $G = C_3 \cup C_5$ and $V(C_3) = \{x_i, i = 0, .., 2\}$, $E(C_3) = \{x_i x_{(i+1)(\text{mod} 3)}, i = 0, .., 2\}$, $V(C_5) = \{y_i, i = 0, ..., 4\}$, $E(C_5) = \{y_i y_{(i+1)(\text{mod} 5)}, i = 0, ..., 4\}$ then $\sigma = (y_1 y_5 x_1 y_4)(y_2 y_3 x_3 x_2)$ is a good permutation of G. If $G = C_4 \cup C_4$ and x_i, $x_i x_{(i+1)(\text{mod} 4)}$, $i = 0, ..., 3$ are vertices and edges, respectively, of the first cycle, x'_i, $x'_i x'_{(i+1)(\text{mod} 4)}$, $i = 0, ..., 3$ are vertices and edges, respectively, of the another cycle then $\sigma = (x_1 x_4 x'_4 x'_1)(x_3 x_2 x'_2 x'_3)$ is a good permutation of G.

It is clear that for every graph of 8 vertices and 8 edges exactly one of possibilities I-VII holds. $\square$

7 Proof

Let G be an embeddable, non exceptional and non special graph of order and size $n \geq 9$. We assume that for $k \leq n$, every embeddable, non exceptional and non special graph of order k and size at least k has a good permutation.

I. There are at least two isolated vertices u_1, u_2 in G.

The vertices u_1, u_2 with any two nonadjacent vertices satisfy condition A of Lemma 8 and with any two adjacent vertices satisfy condition B of Lemma 8. Therefore it is enough to consider the case when every pair of vertices cover at most 5 edges. Let x be a vertex such that $d(x) = \Delta(G)$. It is clear that $\Delta(G) \geq 3$. Observe that in this case there are $n - 3 - d(x)$ vertices nonadjacent to x, each of them of degree at most $5 - d(x)$, and there are $d(x)$ neighbours of x, each of them of degree at most $6 - d(x)$. Calculating the sum of the degrees of the vertices of G we obtain that $2n \leq d(x) + (n - 3 - d(x))(5 - d(x)) + d(x)(6 - d(x))$. Hence $0 \leq (3 - d(x))(n - 5)$ and then $3 \geq d(x) = \Delta(G)$. Since $\Delta(G) = 3$, there are at least four vertices of degree 3. Since every two of them have to be neighbours, there are exactly four vertices of degree 3 and then there are $n - 6$ vertices of degree 2 in G. Therefore G is the union of two isolated vertices, K_4 and the graph G_C, where G_C is the union of cycles. Since G is embeddable, G_C is not a triangle. Let y_1, y_2 be two vertices of degree 3. Then the vertices u_1, y_1, y_2, u_2 satisfy condition B of Lemma 8, cover 5 edges and $G - \{u_i, y_i, i = 1, 2\}$ is embeddable.

II. There is exactly one isolated vertex u and there is at least one vertex v of degree 1 in G.

Let s be the only neighbour of v. For any vertex $x \in V(G) - \{u, v, s\}$, vertices v, s, x, u satisfy condition B of Lemma 8. Therefore it is enough to consider the case when for every vertex $x \notin \{u, v, s\}$, vertices s and x cover at most 5 edges, *i. e.* any vertex nonadjacent to s has degree at most $5 - d(s)$ and any vertex adjacent to s has degree at most $6 - d(s)$. Calculating degrees of vertices we obtain $2n \leq 0 + 1 + d(s) + (n - 2 - d(s))(5 - d(s)) + (d(s) - 1)(6 - d(s))$. Then

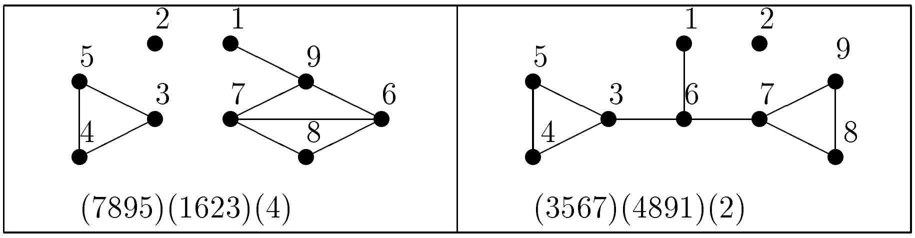


Figure 23: Two (9,9)-graphs with good permutations.

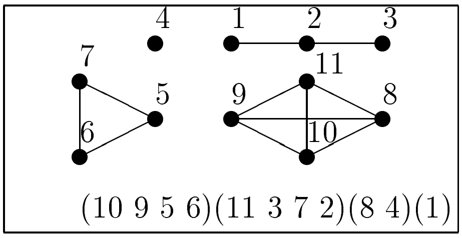


Figure 24: An (11,11)-graph with a good permutation.

$0 \leq (n-5)(3-d(s))$ and hence $3 \geq d(s)$.

Let us suppose that $d(s) = 3$. In this case every neighbour of s, except for v, has degree 3 and every vertex nonadjacent to s, except for u, has degree 2. Let y be a neighbour of s, $y \neq v$. The vertices v, s, x, u cover 5 edges and either $G - \{v, s, x, u\}$ is embeddable or G is isomorphic to one of graphs in Fig. 23.

Let us consider the case when $d(s) = 2$. Let $x \neq v$ be the second neighbour of s. In this case $d(x) \leq 4$ and every vertex from $V(G) - \{u, v, s, x\}$ has its degree at most 3. For every vertex y nonadjacent to x, the vertices y, v, s, x satisfy condition A of Lemma 8. It is enough to consider the case when the mentioned vertices cover at most 5 edges, *i. e.* the neighbours of x has degrees at most 3 and every vertex nonadjacent to x, except for v, has a degree at most $4 - d(x)$. Calculating the sum of the degrees of the vertices we obtain $2n \leq 1 + 2 + d(x) + 3(d(x)-1) + (n-3-d(x))(4-d(x))$. Thus $0 \leq 2n - nd(x) + d^2(x) + 3d(x) - 12$. Since $n \geq 9$, we have $d(x) = 1$. Observe that then there is a vertex y such that $d(y) = 3$. The vertices y, v, s, x cover 5 edges and $G - \{y, v, s, x\}$ is embeddable or G is isomorphic to the graph in Fig. 24.

Let us suppose that $d(s) = 1$. In this case $\Delta(G) \leq 4$. It is clear that $\Delta(G) \geq 3$. Observe that v, s with any two nonadjacent vertices satisfy condition A of Lemma 8. If $\Delta(G) = 4$ then there is a vertex x such that $d(x) = 4$ and, since $n \geq 9$, there is a vertex $y \in V(G) - \{u, v, s\}$ such that y is nonadjacent to x. The vertices x, v, s, y cover at least 6 edges. Let $\Delta(G) = 3$ and let x be a vertex such that $d(x) = 3$. It is easy to check that there is a vertex y nonadjacent to x such that $d(y) \geq 2$. Hence the vertices x, v, s, y cover 6 edges.

III. There is exactly one isolated vertex u and there are no vertices of degree 1 in G.

Calculating degrees of vertices we obtain that either $\Delta(G) = 3$ and there are exactly two vertices a_1, a_2 of degree 3 or $\Delta(G) = 4$, there is exactly one vertex b_1 of degree 4 and there is no vertex of degree 3. It is easy to see that there are three possibilities. If $\Delta(G) = 3$ then either a_1, a_2 are vertices of two disjoint cycles and a_1, a_2 are joined by a path (possibly the edge a_1a_2) or a_1, a_2 are joined by three paths such that a_1, a_2 are the only common vertices of every two of them. If $\Delta(G) = 4$ then b_1 is the only common vertex of two different cycles.

Let G_i, $i = 1, 2, 3$ respectively, denote the component of G containing a_1, a_2 or b_1 in any of just described cases. It is clear that if $|V(G)| > |V(G_i)| + 1$, $i \in \{1, 2, 3\}$ then $G_C = G - V(G_i) - \{u\}$ is the union of disjoint cycles.

Observe that if there are vertices x_1, x_2, x_3 such that either a_1, x_1, x_2, x_3 or b_1, x_1, x_2, x_3 satisfy condition A of Lemma 8 then these vertices cover at least 6 edges. Therefore we may suppose that there are no such three vertices. It is clear that then G_1 and G_3 have cycles of length 3 or 4 and, in G_1, a_1, a_2 are joined by a path of length at most 2. In G_2 the vertices a_1, a_2 are joined by three paths of length 2 or by the edge a_1a_2 and two paths of length at most 3.

It is easy to check that if there is a cycle C of length at least 5 in G_C then four successive vertices of C satisfy condition A of Lemma 8, cover 5 edges and the graph induced by the remaining vertices is embeddable. Therefore let us suppose that every cycle of G_C is of length 3 or 4.

Since $n \geq 9$, only if either G_1 has two cycles of length 4 or G_1 has one cycle of length 4 and a_1, a_2 are joined by a path of length 2 then it is possible that $|V(G_C)| = 0$.

Let us consider the case when there are eight vertices x_i, x'_i, $i = 0, 1, 2$, y_j, $j = 1, 2$ such that $d(x_i) = d(x'_i) = 2$ for $i = 1, 2$, $d(x_0) \geq 2$, $d(x'_0) \geq 2$, $d(y_j) \geq 2$ for $j = 1, 2$, $x_i x_{(i+1)(\mathrm{mod} 3)}$, $x'_i x'_{(i+1)(\mathrm{mod} 3)} \in E(G)$, $i = 0, 1, 2$ (vertices x_i, x'_i respectively, $i = 0, 1, 2$ induce a triangle), $y_1 y_2 \in E(G)$. We permit that only $x_0 x'_0$, $x_0 y_1$, $x'_0 y_2$ can be the remaining edges joining considered vertices. Observe that these vertices cover at least 9 edges. If the vertices x_i, x'_i, for $i = 0, 1, 2$, and y_j, for $j = 1, 2$ cover at least 10 edges then, by Theorem 5, $G' = G - \{x_i, x'_i, y_j; i = 0, 1, 2, j = 1, 2\}$ has a good permutation and $\sigma = (y_1 x'_2 y_2 x_2)(x_0 x_1 x'_0 x'_1)\sigma'$ is a good permutation of G, where σ' is a good permutation of G'. It is easy to check that if the vertices x_i, x'_i, $i = 0, 1, 2$, y_j, $j = 1, 2$ cover 9 edges then G' is embeddable and, by Theorem 5, G' has a good permutation, σ' say. Then σ is a good permutation of G. Therefore we may assume that there are no such eight vertices in G. Observe that this assumption implies that there is at most one triangle in G_C. Moreover if G_1, G_3 have exactly one cycle of length 3 then G_C has no triangles; if G_3 has two cycles of length 3 then either G_C is a triangle or G_C has only cycles of length 4; and also G_1 does not have two different cycles of length 3.

Let us now consider the case when there are eight vertices z_i, z'_i, $i = 0,1,2,3$ such that $z_i z_{(i+1)(\mathrm{mod}4)}$, $z'_i z'_{(i+1)(\mathrm{mod}4)} \in E(G)$, $i = 0,1,2,3$ and one of the following possibilities holds: either $d(z_0) \geq 2$, $d(z'_0) \geq 2$, $d(z_2) \geq 2$, $d(z_1) = d(z'_1) = 2$, $d(z'_2) = 2$, $d(z_3) = d(z'_3) = 2$ or $d(z_0) \geq 2$, $d(z_1) \geq 2$, $d(z_2) = d(z_3) = d(z'_i) = 2$, $i = 0,1,2,3$. We permit that only $z_0 z'_0$ can be the remaining edge joining considered vertices. Observe that these vertices cover at least 8 edges.

If the vertices z_i, z'_i, $i = 0,1,2,3$ cover at least 10 edges then, by Theorem 5 $G' = G - \{z_i, z'_i; i = 0,1,2,3\}$ has a good permutation and $\sigma = (z_0 z_3 z'_2 z'_3)(z_2 z_1 z'_0 z'_1)\sigma'$ is a good permutation of G, where σ' is a good permutation of G'.

If the vertices z_i, z'_i, $i = 0,1,2,3$ cover 9 edges and G' is embeddable then by Theorem 5 G' has a good permutation and σ is a good permutation of G.

Similarly, if the vertices z_i, z'_i, $i = 0,1,2,3$ cover 8 edges and G' is the embeddable, nonexceptional and not special graph then by induction hypothesis G' has a good permutation and σ is a good permutation of G.

Therefore we may assume that there is no layout described above. This implies that if G_2 has two cycles of length 3 then G_C is either the union of a triangle and a cycle of length 4 or the union of one or two cycles of length 4; if G_3 has two cycles of length 3 then G_C is either a triangle or a cycle of length 4; in the remaining cases of G_2 and G_3 G_C is a triangle; if $a_1 a_2 \in E(G)$ then it is possible that G_1 has two cycles of length 4 and in this case G_C is either a triangle or a cycle of length 4; if G_1 has exactly one cycle of length 4 then either $a_1 a_2 \in E(G)$ and G_C is a cycle of length 4 or $a_1 a_2 \notin E(G)$ and $|V(G_C)| = 0$.

Let us suppose finally that there are eight vertices s, t_i, $i = 0,1,2$, w_j, $j = 0,1,2,3$ such that $t_i t_{(i+1)(\mathrm{mod}3)} \in E(G)$, $i = 0,1,2$, $w_j w_{(j+1)(\mathrm{mod}4)} \in E(G)$, $j = 0,1,2,3$ and one of the two following possibilities holds:

i) either $d(s) \geq 0$, $d(t_2) \geq 2$, $d(w_0) \geq 2$, $d(w_1) \geq 2$, $d(t_0) = d(t_1) = d(w_2) = d(w_3) = 2$ and we permit that only st_2, sw_1 can be the remaining edges joining considered vertices

ii) or $d(s) \geq 0$, $d(t_0) \geq 2$, $d(t_2) \geq 2$, $d(w_2) \geq 2$, $d(t_1) = d(w_0) = d(w_1) = d(w_3) = 2$ and we permit that only st_2, sw_2 can be the remaining edges joining considered vertices.

Set $G' = G - \{s, t_i, w_j; i = 0,1,2, j = 0,1,2,3\}$. If either the eight vertices cover

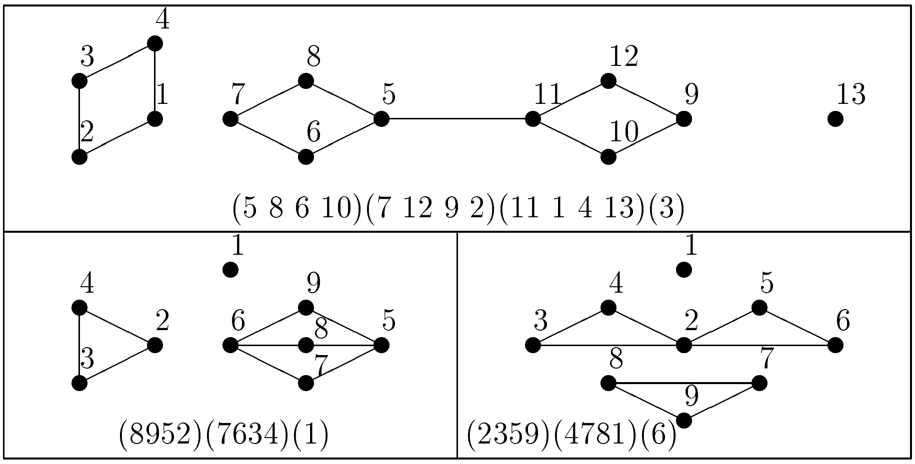


Figure 25: An (13,13)-graph and two (9,9)-graphs with good permutations.

at least 10 edges or the eight vertices cover 9 edges and G' is embeddable or the eight vertices cover 8 edges and G' is the embeddable, nonexceptional, nonspecial graph then, by Theorem 5 or by induction hypothesis, respectively, G' has a good permutation σ' and $\sigma = (w_0t_0w_1w_2)(w_3st_1t_2)\sigma'$ is a good permutation of G. Therefore we may assume that there are no such eight vertices. This assumption implies that G is one of the graphs in Fig. 25 .

IV. There are no isolated vertices and there are two vertices v_1, $v_2 \in V(G)$ such that v_1v_2 is an isolated edge.

It is easily seen that $\Delta(G) \geq 3$. Let $x_1 \in V(G)$ such that $d(x_1) = \Delta(G)$. If x_1 is the neighbour of every vertex from $V(G) - \{v_1, v_2\}$ then there is a vertex $x_2 \notin \{v_1, v_2\}$ such that $d(x_2) = 1$ and hence the vertices v_1, v_2, x_1, x_2 satisfy condition B of Lemma 8 and cover at least 6 edges. Let us suppose that x_1 is not adjacent to every vertex of $V(G) - \{v_1, v_2\}$. Observe that there is a vertex $x_3 \notin \{v_1, v_2\}$ nonadjacent to x_1 such that if $d(x_1) \geq 4$ then $d(x_3) \geq 1$ and if $d(x_1) = 3$ then, since $n \geq 9$, $d(x_3) \geq 2$. Thus the vertices x_1, v_1, v_2, x_3 satisfy the condition A of Lemma 8 and cover at least 6 edges.

V. There is neither isolated vertex nor isolated edge in G and there are two vertices v_1, v_2 of degree 1 such that $N(v_1) \neq N(v_2)$.

Let y_i be the only neighbour of v_i for $i = 1, 2$. Since the vertices v_1, y_1, y_2, v_2 satisfy condition B of Lemma 8, it is enough to consider the case when the vertices y_1, y_2 cover at most 5 edges. Let c be the number of edges covered by y_1 or y_2. It is clear that $c \geq 3$ and that without loss of generality we may assume that for every pair of vertices v_p, v_q of degree 1 such that $\{y_q\} = N(v_q) \neq N(v_p) = \{y_p\}$, the vertices y_p and y_q cover at most c edges.

Let us first consider the case when $d(y_2) \geq d(y_1) = 2$. By t_1 we shall denote the second neighbour of y_1. Observe that since $c \leq 5$, it is impossible that t_1 is a neighbour of every vertex from $V(G) - \{v_1\}$.

For any vertex $x \neq v_1$ nonadjacent to t_1 the vertices x, v_1, y_1, t_1 satisfy condition A of Lemma 8. If $d(t_1) \geq 4$ then for any vertex $x \neq v_1$ nonadjacent to t_1 the vertices x, v_1, y_1, t_1 cover at least 6 edges and, by Theorem 5, G has a good permutation.

Let us consider the case when $d(t_1) = 3$. If there is a vertex $x \neq v_1$ nonadjacent to t_1 such that $d(x) \geq 2$ then, by Theorem 5, G has a good permutation. Let us suppose that every vertex nonadjacent to t_1 is of degree 1. Since there are no isolated edges, every vertex of G, except for v_1, is a neighbour of t_1 or w_1 or w_2, where w_1, w_2 are the remaining neighbours of t_1. Since $c \leq 5$, $d(w_i) \leq 3$ for $i = 1, 2$ and either at least one of vertices w_1, w_2 has a degree 1 or if $d(w_i) \geq 2$ for $i = 1, 2$, then $d(w_1) + d(w_2) \leq 5$. In both cases $|V(G)| \leq 8$.

Let us consider the case when $d(t_1) = 2$. By w_1 we shall denote the second neighbour of t_1. If there is a vertex $x \neq w_1$, $d(x) \geq 3$ then the vertices x, v_1, y_1, t_1 cover at least 6 edges and, by Theorem 5, G has a good permutation. If $d(w_1) \geq 4$ then the vertices v_1, y_1, t_1, w_1 satisfy condition A of Lemma 8, cover at least 6 edges and hence G has a good permutation. So we may assume that $d(w_1) \leq 3$ and $d(x) \leq 2$ for any $x \in V(G) - \{v_1, y_1, t_1, w_1\}$. Calculating the degrees of the vertices of G we obtain that if $d(w_1) = 1$ then $2n = 2 \mid E(G) \mid \leq d(v_1) + d(y_1) + d(t_1) + d(w_1) + 2(n-4) = 2n - 2$ and if $d(w_1) \geq 2$ then $d(v_1) + d(y_1) + d(t_1) + d(w_1) + d(v_2) + 2(n-5) = 2n + d(w_1) - 4$, which is impossible.

Let us consider the case when $d(t_1) = 1$. If $d(y_2) = 2$ then without loss of generality we may assume that $d(t_2) = 1$, where t_2 is the second neighbour of y_2. Observe that there is a vertex x such that $d(x) \geq 3$. The vertices x, v_1, y_1, t_1 cover at least 5 edges and $G - \{x, v_1, y_1, t_1\}$ is always embeddable. Therefore, by Theorem 5 and Lemma 8, G has a good permutation. Let us suppose that $d(y_2) = 3$. Since the vertices y_1, v_1, v_2, y_2 satisfy condition A of Lemma 8 and cover 5 edges, it is enough to consider the case when $G - \{v_i, y_i; i = 1, 2\}$ is not embeddable. In this case G is isomorphic to one of four graphs: two of order 9 and two of order 11. In each of these graphs there is a vertex $x \neq y_2$ such that $d(x) = 3$. The vertices x, v_1, y_1, t_1 satisfy condition A of Lemma 8, cover 5 edges and $G - \{v_1, y_1, t_1, x\}$ is embeddable.

Let us finally consider the case when $d(y_i) = 3$ for $i = 1, 2$. Since $c \leq 5$, vertices y_1, y_2 are adjacent. Without loss of generality we may assume that for any vertex v of degree 1 if y is the neighbour of v then $d(y) = 3$ and hence y is adjacent to y_1 and y_2. The vertices v_1, y_1, y_2, y_2 satisfy condition B of Lemma 8 and cover 5 edges. Therefore it is enough to consider the case when $G - \{v_i, y_i; i = 1, 2\}$ is not embeddable. Then G is isomorphic to one of graphs in Fig. 26.

VI. There is no isolated vertex in G, there is at least one vertex of de-

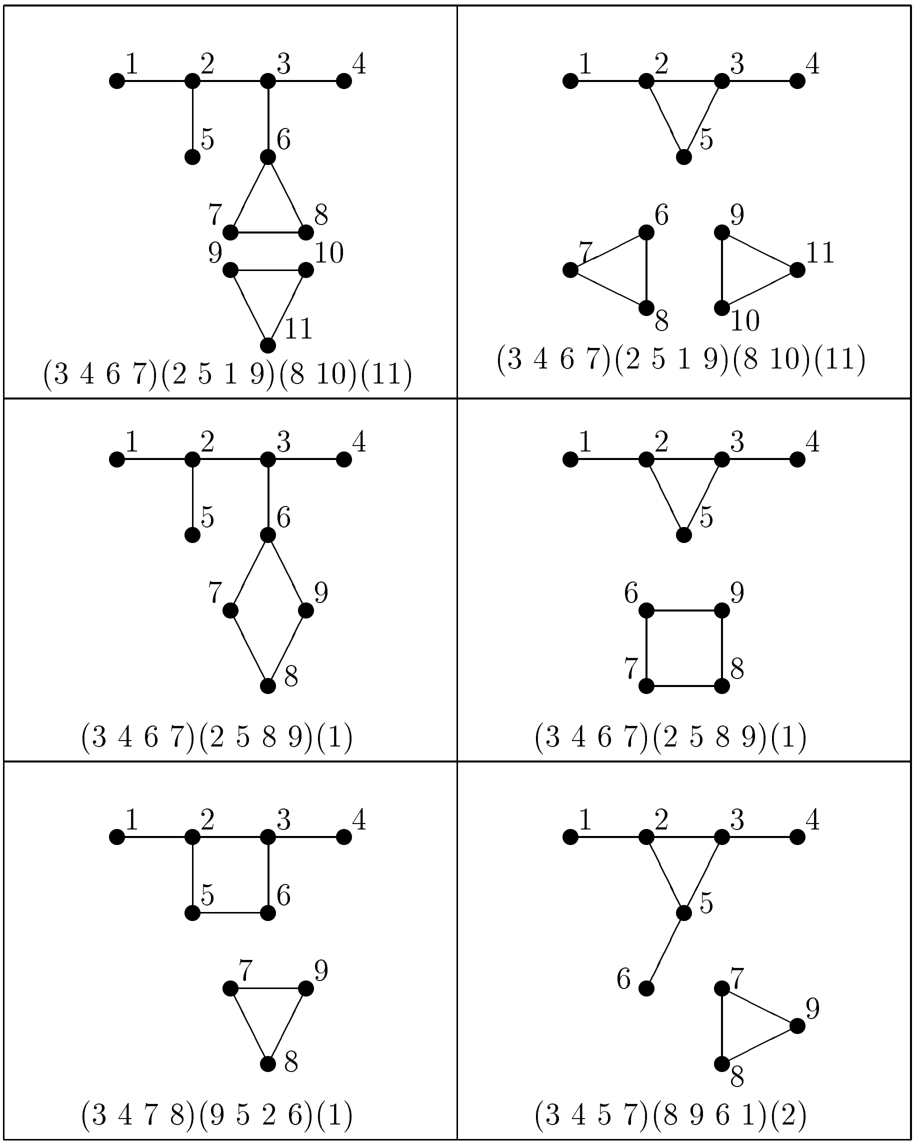


Figure 26: Two (11,11)-graphs and four (9,9)-graphs with good permutations.

gree 1 in G and all the vertices of degree 1 have a common neighbour w, such that $d(w) \geq 1$.

It is obvious that $\Delta(G) \leq n-1$. Set $L_i = \{u \in V(G) - \{w\} : d(u) = i\}$ and $l_i = |L_i|$ for $i = 1, ..., n-1$. Write $r = d(w) - l_1$. Calculating degrees of vertices we obtain that $2n = (r + l_1) + \sum_{i=1}^{n-1}(il_i)$. Since $\sum_{i=1}^{n-1} l_i = n-1$, we have $2 = r + \sum_{i=3}^{n-1} l_i(i-2)$ and hence $l_i = 0$ for $i \geq 5$. Therefore $2 = r + l_3 + 2l_4$ and there are four possible subcases:

(a) $l_4 = 1$, $l_3 = r = 0$,

(b) $l_4 = 0$, $l_3 = 2$, $r = 0$,

(c) $l_4 = 0$, $l_3 = r = 1$,

(d) $l_4 = l_3 = 0$, $r = 2$.

Subcases (a) and (b) It is clear that $l_1 \geq 2$. Observe that after removing the vertices of degree 1 from G we obtain the graph with exactly one isolated vertex w. Such graphs are described in section III. We can use the same notation as in section III only replacing the isolated vertex u by the vertex w. We can also consider successively the same layouts of vertices as in section III. It is clear that here in more cases it is possible that $|V(G_C)| = 0$. As in section III we may assume that G_1 and G_3 have cycles of length 3 or 4. In G_1 vertices a_1, a_2 are joined by a path of length at most 2. In G_2 vertices a_1, a_2 are joined by three paths of length 2 or by the edge a_1a_2 and two paths of length at most 3. We may also assume that every cycle of G_C is of length 3 or 4. Next there are some differences.

The assumption that there are no eight vertices x_i, x'_i, $i = 0, 1, 2$, y_j, $j = 1, 2$, described in III, such that $G - \{x_i, x'_i, y_j; i = 0, 1, 2, j = 1, 2\}$ is embeddable implies that there is at most one triangle in G_C and moreover: G_1 does not have two different cycles of length 3, if G_1 has exactly one cycle of length 3 then G_C has no triangles and if G_3 has at least one cycle of length 3 then G_C has no triangles.

The assumption that there are no eight vertices z_i, z'_i, $i = 0, 1, 2, 3$, described in III, such that $G - \{z_i, z'_i; i = 0, ..., 3\}$ has a good permutation implies that:
- if G_2 or G_3 has two cycles of length 3 then G_C has at most one cycle of length 4 and in the remaining cases of G_2, G_3 the graph G_C has no cycles of length 4,
- in G_1 if a_1, a_2 are joined by a path of length 2 then G_1 does not have two different cycles of length 4 and G_C has no cycles of length 4,
- in G_1 with $a_1a_2 \in E(G)$ and with two cycles of length 4 we have that either $|V(G_C)| = 0$ or G_C is a triangle and $d(w) \geq 4$.
- in G_1 with $a_1a_2 \in E(G)$ and with exactly one cycle of length 4 either $d(w) < 4$ and G_C has no cycle of length 4 or $d(w) \geq 4$ and G_C is a cycle of length 4.

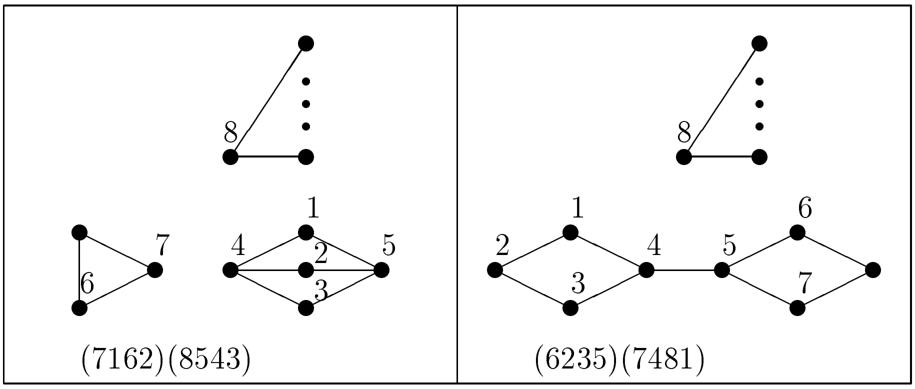


Figure 27: Two (n,n)-graphs.

The assumption that there are no eight vertices s, t_i, $i = 0,1,2$, w_j, $j = 0,1,2,3$, described in III, such that $G - \{s, t_i, w_j; i = 0,1,2, j = 0,...,3\}$ has a good permutation implies that:
- in G_1 we obtain that $a_1a_2 \in E(G)$ and $|V(G_C)| = 0$,
- in G_2 if either vertices a_1, a_2 are joined by three paths of length 2 or by the edge a_1a_2 and two paths of length 2 then either $|V(G_C)| = 0$ or G_C is a triangle,
- in the remaining cases of G_1, G_3 we obtain $|V(G_C)| = 0$.

There are two possibilities. The first one is that it is possible to obtain a graph of order 8 from G by removing vertices of degree 1 (in G). It is easy to check that every obtained graph of order 8 has a good permutation. It is clear that in these good permutations the vertex w is not a fixed point. Hence we can extend such a good permutation on 8 vertices to a good permutation of G such that the set of removed vertices is mapped on itself in proper way. The other possibility is that G is isomorphic to one of graphs in Fig. 27. Observe that the permutations on 8 vertices indicated in the figure can be extend to a good permutation of G such that the set of unlabeled vertices is mapped on itself in proper way.

Subcase (c) Let a_1 be the vertex of degree 3 such that $a_1 \neq w$.
Observe that the vertices a_1 and w are joined by a path (possibly the edge a_1w) and a_1 is a vertex of a cycle. By G_S we shall denote the component of G such that $a_1 \in V(G_S)$. If $|V(G)| > |V(G_S)|$ then $G_C = G - V(G_S)$ is a union of cycles. If there are vertices x_i, $d(x_i) \geq 2$, $i = 1,2,3$ such that a_1, x_1, x_2, x_3 satisfy condition A of Lemma 8 then these four vertices cover 6 edges. Therefore we may assume that there are no such three vertices. This assumption implies that the cycle of G_S has the length at most 4 and that the path joining vertices a_1 and w has the length at most 2. It is clear that since G is embeddable, it is possible that $|V(G)| = |V(G_S)|$ when the cycle of G_S has the length 4. Next we

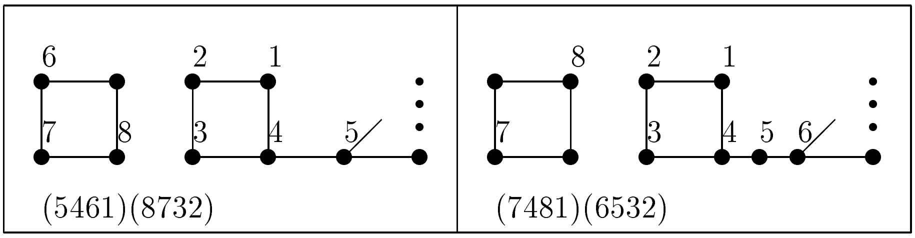


Figure 28: Two (n,n)-graphs.

shall consider the same layouts of vertices as in section III. Similarly as in III we may assume that every cycle of G_C is of length 3 or 4.

The assumption that there are no eight vertices x_i, x'_i, $i = 0, 1, 2$, y_j, $j = 1, 2$, described in III, such that $G - \{x_i, x'_i, y_j; i = 0, 1, 2, j = 1, 2\}$ has a good permutation implies that there is at most one triangle in G_C and, moreover, if G_S has a triangle then either G_C does not have triangles or G_C is the union of a triangle and one cycle of length 4 with $d(w) = 3$ for $a_1w \in E(G)$ and with $d(w) = 2$ for $a_1w \notin E(G)$.

The assumption that there are no eight vertices z_i, z'_i, $i = 0, 1, 2, 3$, described in III, such that $G - \{z_i, z'_i; i = 0, ..., 3\}$ has a good permutation implies that:
- if G_S has a cycle of length 4 then either G_C does not have cycles of length 4 or G_C is a cycle of length 4 or else G_C is the union of a triangle and a cycle of length 4 with $d(w) \geq 5$ when $a_1w \in E(G)$ and with $d(w) \geq 4$ when $a_1w \notin E(G)$,
- for G_S with a triangle we obtain that if $d(w) \leq 3$ and $a_1w \notin E(G)$ then G_C has at most one cycle of length 4.
- in the remaining cases of G_S with a triangle we obtain that G_C has at most two cycles of length 4 .

The assumption that there are no eight vertices s, t_i, $i = 0, 1, 2$, w_j, $j = 0, 1, 2, 3$, described in III, such that $G - \{s, t_i, w_j; i = 0, 1, 2, j = 0, ..., 3\}$ has a good permutation implies that G_S has the cycle of length 4 and either $|V(G_C)| = 0$ or G_C is a cycle of length 4. If $|V(G_C)| = 0$ then we may obtain a graph of order 8 from G by removing vertices of degree 1 (in G). Every obtained graph of order 8 has a good permutation. Since the vertex w is not a fixed point of any of them then we can extend such a good permutation on 8 vertices to a good permutation of G. If G_C is a cycle of length 4 then G is isomorphic to one of graphs on Fig. 28. Observe that the permutations on 8 vertices indicated in the figure can be extended to a good permutation of G.

Subcase (d) It is clear that after removing vertices of degree 1 from G we obtain the union of cycles. By G_S we shall denote the component of G such that $w \in V(G_S)$. Set $G_C = G - V(G_S)$. If there are vertices x_i, $i = 1, 2, 3$ such that w, x_1, x_2, x_3 satisfy condition A of Lemma 8 then these four vertices cover 6 edges. Therefore we may assume that there are no such three vertices. This assumption implies that the cycle of G_S has the length at most 4. Since G is embeddable, $|V(G)| > |V(G_S)|$. Next we shall consider the same layouts of vertices as in section III. Similarly as in III we may assume that every cycle of G_C is of length 3 or 4. Since G is embeddable, if G_S has a triangle then G_C is not a triangle.

The assumption, described in III, that there are no eight vertices x_i, x_i', $i = 0, 1, 2$, y_j, $j = 1, 2$, such that $G - \{x_i, x_i', y_j; i = 0, 1, 2, j = 1, 2\}$ has a good permutation implies that there is at most one triangle in G_C.

The assumption, described in III, that there are no eight vertices z_i, z_i', $i = 0, 1, 2, 3$, such that $G - \{z_i, z_i'; i = 0, ..., 3\}$ has a good permutation implies that if G_S has a triangle then there are at most two cycles of length 4 in G_C and if G_S has a cycle of length 4 then either G_C is a triangle or $d(w) = 3$ and G_C is the union of a triangle and a cycle of length 4.

The assumption, described in III, that there are no eight vertices s, t_i, $i = 0, 1, 2$, w_j, $j = 0, 1, 2, 3$, such that $G - \{s, t_i, w_j; i = 0, 1, 2, j = 0, ..., 3\}$ has a good permutation implies that if G_S has a triangle then G_C is a cycle of length 4 and if G_S has a cycle of length 4 then G_C is a triangle. Then we are able to obtain a graph of order 8 from G by removing vertices of degree 1. Every obtained graph of order 8 has a good permutation and we can extend such a good permutation on 8 vertices to a good permutation of G.

VII. The graph G is the union of cycles.

Let C_l denote a cycle of length l, $l \geq 3$. If there is a cycle of length at least 5 then its four successive vertices satisfy the condition A of Lemma 8 and cover 5 edges. Therefore it is enough to consider the case when the subgraph of G induced by the remaining vertices is not embeddable. Hence we may assume that either there is no cycle of length at least 5 or G is one of te following graphs:

- $G = C_5 \cup 2C_3$
- $G = C_5 \cup C_4$
- $G = C_6 \cup C_3$.

- If $G = C_5 \cup 2C_3$ then $(a_0a_1a_3b_0)(b_1b_2c_1c_2)(a_2c_0)(a_4)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k; i = 0, ..., 4, j = 0, 1, 2, k = 0, 1, 2\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}5)}, b_jb_{(j+1)(\text{mod}3)}, c_kc_{(k+1)(\text{mod}3)}; i = 0, ..., 4, j = 0, 1, 2, k = 0, 1, 2\}$.

- If $G = C_5 \cup C_4$ then $(a_0a_1a_3b_0)(a_2b_2b_3b_1)(a_4)$ is a good permutation of G, where $V(G) = \{a_i, b_j; i = 0, ..., 4, j = 0, ..., 3\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}5)}, b_jb_{(j+1)(\text{mod}4)}; i = 0, ..., 4, j = 0, 1, 2, 3\}$.

- If $G = C_6 \cup C_3$ then $(a_0a_1a_3b_2)(b_0b_1a_2a_5)(a_4)$ is a good permutation of G, where $V(G) = \{a_i, b_j; i = 0, ..., 5, j = 0, 1, 2\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}6)}, b_jb_{(j+1)(\text{mod}3)}; i = 0, ..., 5, j = 0, 1, 2\}$.

Hence we may assume that every cycle of G has length 3 or 4. Next we shall consider the same layouts of vertices as in section III. The assumption that there are no eight vertices z_i, z_i', $i = 0, 1, 2, 3$, described in III, such that $G - \{z_i, z_i'; i = 0, 1, ..., 3\}$ has a good permutation implies that either G has at most one cycle of length 4 or $G = 2C_4 \cup 2C_3$ or $G = 3C_4$ or else $G = 2C_4 \cup C_3$. Since $n \geq 9$, if G has one cycle of length 4 then G has at least two triangles and if G has no cycles of length 4 then G has at least three triangles. Assumption that there are no eight vertices x_i, x_i', $i = 0, 1, 2$, y_j, $j = 1, 2$, described in III, such that $G - \{x_i, x_i', y_j; i = 0, 1, 2, j = 1, 2\}$ has a good permutation implies that either $G = C_4 \cup 2C_3$ or $G = C_4 \cup 3C_3$ or $G = 5C_3$ or $G = 4C_3$ or $G = 3C_4$ or else $G = 2C_4 \cup C_3$.

- If $G = C_4 \cup 2C_3$ then $(a_0a_1b_0b_1)(a_2a_3c_0c_1)(b_2c_2)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k; i = 0, ..., 3, j = 0, 1, 2, k = 0, 1, 2\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}4)}, b_jb_{(j+1)(\text{mod}3)}, c_kc_{(k+1)(\text{mod}3)}; i = 0, ..., 3, < cr > j = 0, 1, k = 0, 1, 2\}$.

- If $G = C_4 \cup 3C_3$ then $(a_0a_1b_0b_1)(a_2a_3c_0c_1)(d_0c_2d_1b_2)(d_2)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k, d_l; i = 0, ..., 3, j, k, l = 0, 1, 2\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}4)}, b_jb_{(j+1)(\text{mod}3)}, c_kc_{(k+1)(\text{mod}3)}, d_ld_{(l+1)(\text{mod}3)}; i = 0, ..., 3, j = 0, 1, 2, k = 0, 1, 2, l = 0, 1, 2\}$.

- If $G = 5C_3$ then $(a_0a_1b_0b_1)(a_2c_0c_1d_0)(b_2c_2e_0e_1)(d_1e_2)(d_2)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k, d_l, e_m; i, j, k, l, m \in \{0, 1, 2\}\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}3)}, b_jb_{(j+1)(\text{mod}3)}, c_kc_{(k+1)(\text{mod}3)}, d_ld_{(l+1)(\text{mod}3)}$,
$e_me_{(m+1)(\text{mod}3)}$,
$i, j, k, l = 0, 1, 2\}$.

- If $G = 4C_3$ then $(a_0a_1b_0b_1)(a_2c_0c_1d_0)(c_2b_2d_1d_2)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k, d_l; i, j, k, l \in \{0, 1, 2\}\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}3)}, b_jb_{(j+1)(\text{mod}3)}, c_kc_{(k+1)(\text{mod}3)}, d_ld_{(l+1)(\text{mod}3)}; i, j, k, l = 0, 1, 2\}$.

- If $G = 3C_4$ then $(a_0a_1b_0b_1)(a_2a_3c_0c_1)(b_2b_3c_2c_3)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k; i, j, k = 0, ..., 3\}$, $E(G) = \{a_ia_{(i+1)(\text{mod}4)}, b_jb_{(j+1)(\text{mod}4)}, c_kc_{(k+1)(\text{mod}4)}; i, j, k = 0, ..., 3\}$.

- If $G = 2C_4 \cup C_3$ then $(a_0a_1b_0b_1)(a_2a_3c_0c_1)(b_2c_2)(b_3)$ is a good permutation of G, where $V(G) = \{a_i, b_j, c_k; i, j \in \{0, ..., 3\}, k = 0, 1, 2\}$ $E(G) =$

$\{a_i a_{(i+1)(\mathrm{mod}4)}, b_j b_{(j+1)(\mathrm{mod}4)}, c_k c_{(k+1)(\mathrm{mod}3)}; i = 0, ..., 3, j = 0, ..., 3, k = 0, 1, 2, \}$.

It is clear that for every embeddable, nonspecial and nonexceptional graph of order and size n exactly one of possibilities I - VII holds. ■

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