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Regular Sets of Descendants by Leftmost Strategy

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Abstract

For a constructor-based rewrite system R, a regular set of ground terms E, and assuming some additional restrictions, we build a finite tree automaton that recognizes the descendants of E, i.e. the terms issued from E by rewriting, according to leftmost strategy.

1 Introduction

Reduction strategies in rewriting and programming have drawn an increasing attention within the last years, and matter both from a theoretical point of view, if the computation result is not unique, and from a practical point of view, for termination and efficiency. For a leftmost-strategy $left$, expressing by a finite tree automaton the $left$-descendants $R^*_{left}(E)$ of E, can help to study $left$-strategy: in particular it allows to decide $left$-reachability since $t_1 \ {}^{left}\!\!\rightarrow^* t_2 \iff t_2 \in R^*_{left}(\{t_1\})$, and $left$-joinability since $t_1 \overset{left}{\downarrow} t_2 \iff R^*_{left}(\{t_1\}) \cap R^*_{left}(\{t_2\}) \neq \emptyset$. More generally, it can help with the static analysis of rewrite programs, and by extension, of functional programs. Computations of innermost, innermost-leftmost and outermost descendants have been investigated in [2] by the authors.

We build a finite automaton that can express the sets of leftmost descendants of E with respect to a constructor-based rewrite system, assuming:

1. E is the set of ground constructor-instances (also called data-instances) of a given linear term t (i.e. $E = \{t\theta\}$).
2. Every rewrite rule is linear (both sides).
3. In right-hand-sides, there are no nested defined-functions, and arguments of defined-functions are either variables or ground terms.
4. Permutative rewrite rules are forbidden.
5. Every rewrite rule is variable-preserving.
6. In a rhs, a variable x located below a constructor does not occur on the left of a defined-function.[1]

It is shown in [1] that if any restriction among 1, 2, 3 is not satisfied, then the set of descendants without a strategy is not regular.

Compared to the innermost-leftmost descendants of [2], we assume more restrictions (4, 5, 6). This due to the fact that computing leftmost descendants is more difficult than computing only innermost-leftmost ones. However, unlike [2],

[1] For example, $c(x, f(y))$ is forbidden if c is a constructor and f a function.

rhs's containing several defined-functions are allowed, thanks to the new result of section 3.

The paper is organized as follows. Section 3 gives the computation of the language $R_p^{*\lhd}(E)$ (i.e. terms obtained after rewriting eventually at position p of t plus possibly at functions positions issued from rhs's, according to a leftmost strategy). Leftmost descendants are computed in section 4 (we apply the previous language for every p position of t, by checking out before that sub-terms occurring on the left of p be normalized by intersection with an automaton that recognized irreducible terms). Now, let us see, in the following section preliminaries notions. Subsection 2.1 (usual notions) may be skipped by the reader.

2 Preliminaries

2.1 Usual notions: term rewriting and tree automata

Let C be a finite set of *constructors* and F be a finite set of *defined-function symbols* (*functions* in a shortened form). For $c \in C \cup F$, $ar(c)$ is the arity of c. *Terms* are denoted by letters s, t. A *data-term* is a *ground* term (i.e. without variables) that contains only constructors. T_C is the set of data-terms, and $T_{C\cup F}$ the set of ground terms that contains constructors and functions. For a term t, $Var(t)$ is the set of variables appearing in t, $Pos(t)$ is the set of *positions* of t, $\overline{Pos}(t)$ is the set of non-variable positions of t, $PosF(t)$ is the set of defined-function positions of t. $PosVar(t)$ is the set of variable positions of t. For $p \in Pos(t)$, $t|_p$ is the subterm of t at position p, $t(p)$ is the top symbol of $t|_p$, and $t[t']_p$ denotes the subterm replacement. For positions p, p', $p \geq p'$ means that p is located below p', i.e. $p = p'.v$ for some position v, whereas $p \| p'$ means that p and p' are incomparable, i.e. $\neg(p \geq p') \wedge \neg(p' \geq p)$. The term t contains *nested functions* if there exist $p, p' \in PosF(t)$ s.t. $p > p'$. The domain $dom(\theta)$ of a substitution θ is the set of variables x s.t. $x\theta \neq x$.

A *rewrite rule* is an oriented pair of terms, written $l \to r$. A *rewrite system* R is a finite set of rewrite rules. *lhs* stands for left-hand-side, *rhs* for right-hand-side. R is *constructor-based* if every lhs l of R is of the form $l = f(t_1, \ldots, t_n)$ where $f \in F$ and $t_1, \ldots, t_n$ do not contain any functions. The rewrite relation $\to_R$ is defined as follows: $t \to_R t'$ if there exist $p \in Pos(t)$, a rule $l \to r \in R$, and a substitution θ s.t. $t|_p = l\theta$ and $t' = t[r\theta]_p$ (also denoted by $t \to_{[p, l\to r, \theta]} t'$). $\to_R^*$ denotes the reflexive-transitive closure of $\to_R$. t is *irreducible* if $\neg(\exists t' \mid t \to_R t')$. $t \to_{[p]} t'$ is *leftmost* if $\forall v$ occurring strictly on the left of p, $t|_v$ is irreducible.

A (bottom-up) finite tree *automaton* is a quadruple $\mathcal{A} = (C \cup F, Q, Q_f, \Delta)$ where $Q_f \subseteq Q$ are sets of states and Δ is a set of *transitions* of the form $c(q_1, \ldots, q_n) \to q$ where $c \in C \cup F$ and $q_1, \ldots, q_n, q \in Q$, or of the form $q_1 \to q$ (*empty transition*). Sets of *states* are denoted by letters Q, S, D, and states by q, s, d. $\to_\Delta$ (also denoted $\to_\mathcal{A}$) is the rewrite relation induced by Δ. A ground term t is *recognized* by $\mathcal{A}$ into q if $t \to_\Delta^* q$. $L(\mathcal{A})$ is the set of terms recognized by $\mathcal{A}$ into any states of Q_f. A derivation $t \to_\Delta^* q$ where $q \in Q_f$ is called a *successful run* on t. The states of Q_f are called *final states*. A *Q-substitution* σ is a substitution s.t. $\forall x \in dom(\sigma)$, $x\sigma \in Q$. A set E of ground terms is regular if there exists a finite automaton $\mathcal{A}$ s.t. $E = L(\mathcal{A})$.

2.2 Nesting Automata and Discrimination

Intuitively, the automaton $\mathcal{A}$ discriminates position p into state q means that along every successful run on $t \in L(\mathcal{A})$, $t|_p$ (and only this subterm) is recognized into q. This property allows to modify the behavior of $\mathcal{A}$ below position p without modifying the other positions, by replacing $\mathcal{A}|_p$ by another automaton $\mathcal{A}'$. See [1] for proofs of Lemmas 2.2 and 2.4, and [2] for the proof of Lemma 2.5.

Definition 2.1 *The automaton* $\mathcal{A} = (C \cup F, Q, Q_f, \Delta)$ discriminates *the position* p *into the state* q *if*

- $L(\mathcal{A}) \neq \emptyset$,
- *and* $\forall t \in L(\mathcal{A}),\ p \in Pos(t)$,
- *and for each successful derivation* $t \to_\Delta^* t[q']_{p'} \to_\Delta^* q_f$ *where* $q_f \in Q_f$, *we have*
    - $q' \to_\Delta^* q$ *(i.e. by empty transitions) if* $p' = p$,
    - $q' \neq q$ *otherwise.*

In this case we define the automaton $\mathcal{A}|_p = (C \cup F, Q, \{q\}, \Delta)$.

Lemma 2.2 *Then* $L(\mathcal{A}|_p) = \{t|_p \mid t \in L(\mathcal{A})\}$.

Definition 2.3 *Let* $\mathcal{A} = (C \cup F, Q, Q_f, \Delta)$ *be an automaton that discriminates the position* p *into the state* q, *and let* $\mathcal{A}' = (C \cup F, Q', Q'_f, \Delta')$ *s.t.* $Q \cap Q' = \emptyset$. *We define*

$$\begin{aligned}\mathcal{A}[\mathcal{A}']_p = (C \cup F,\ Q \cup Q',\ Q_f,\quad & \Delta \backslash \{l \to r \mid l \to r \in \Delta \,\wedge\, r = q\} \\ & \cup \Delta' \cup \{q'_f \to q \mid q'_f \in Q'_f\})\end{aligned}$$

Lemma 2.4 $L(\mathcal{A}[\mathcal{A}']_p) = \{t[t']_p \mid t \in L(\mathcal{A}),\, t' \in L(\mathcal{A}')\}$, *and* $\mathcal{A}[\mathcal{A}']_p$ *still discriminates* p *into* q. *Moreover, if* $\mathcal{A}$ *discriminates another position* p' *s.t.* $p' \not\geq p$, *into the state* q', *then* $\mathcal{A}[\mathcal{A}']_p$ *still discriminates* p' *into* q'.

Lemma 2.5 *Let* $\mathcal{A}, \mathcal{B}$ *be automata, and let* $\mathcal{A} \cap \mathcal{B}$ *be the classical automaton used to recognize intersection, whose states are pairs of states of* $\mathcal{A}$ *and* $\mathcal{B}$.
If $\mathcal{A}$ *discriminates* p *into* $q_\mathcal{A}$, $\mathcal{B}$ *discriminates* p *into* $q_\mathcal{B}$, *and* $L(\mathcal{A}) \cap L(\mathcal{B}) \neq \emptyset$, *then* $\mathcal{A} \cap \mathcal{B}$ *discriminates* p *into* $(q_\mathcal{A}, q_\mathcal{B})$.

2.3 Positions and Antecedents

Given a term t, we define :

Definition 2.6 *Let* $p \in Pos(t)$. $Succ(p)$ *are the nearest function positions below* p *:*

$$Succ(p) = \{p' \in PosF(t) \mid p' > p \text{ and } \forall q \in Pos(t)\ (p < q < p' \Rightarrow q \notin PosF(t))\}$$

Definition 2.7 *Let* $p, p' \in Pos(t)$. $p \lhd p'$ *means that* p *occurs strictly on the left of* p', *i.e.* $p = u.i.v$, $p' = u.i'.v'$, *where* $i, i' \in I\!N$ *and* $i < i'$.

Definition 2.8 *Let* $t \to_{[q, l \to r]} t'$ *be a rewrite step.*
Let $v' \in Pos(t')$. v *is an antecedent of* v' *in* t *(denoted* $ant(v', t)$*) through this step if:*

1. $v \in Pos(t)$
2. - $v' \lhd q$ *and* $v = v'$ *or,*
   - $\exists p' \in PosVar(r)$ *with* $r|_{p'} = x$ *s.t.* $v' = q.p'.w$ *and* $v = q.p''.w$ *where* p'' *is a position of* x *in* l.

Remark : *Since lhs's are linear, the antecedent (if exists) is unique.*

Definition 2.9 *Let the following derivation:*

$$t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow \dots t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1} \qquad (1)$$

$v_0 \in Pos(t_0)$ *is an antecedent of* $v_{n+1} \in Pos(t_{n+1})$ *through this derivation if* $\exists v_1 \in pos(t_1), \dots, v_n \in Pos(t_n)$ *s.t.* $\forall i \in \{0, ..., n\}$, v_i *is an antecedent of* v_{i+1} *through step* $t_i \rightarrow t_{i+1}$.

2.4 Descendants and irreducible ground terms

t' is a *descendant* of t if $t \rightarrow_R^* t'$. t' is a *normal-form* of t if $t \rightarrow_R^* t'$ and t' is irreducible. If E is a set of ground terms, $R^*(E)$ denotes the set of descendants of elements of E. $IRR(R)$ denotes the set of irreducible ground terms. $R_{left}^*(E)$ denotes the set of descendants of E, according to a leftmost strategy.

Definition 2.10 $t \rightarrow_{[p, rhs's]}^+ t'$ *means that* t' *is obtained by rewriting* t *at position* p*, plus possibly at positions coming from the rhs's.*
Formally, there exist some intermediate terms $t_1, \dots, t_n$ *and some sets of positions* $P(t), P(t_1)$,
$\dots, P(t_n)$ *s.t.*

$$t = t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow_{[p_1, l_1 \rightarrow r_1]} \dots \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}]} t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1} = t'$$

where

- $p_0 = p$ *and* $P(t) = \{p\}$,
- $\forall j,\ p_j \in P(t_j)$,
- $\forall j,\ P(t_{j+1}) = P(t_j) \backslash \{p' \mid p' \geq p_j\} \cup \{p_j.w \mid w \in PosF(r_j)\}$.

Remark : $P(t_j)$ contains only function positions. Since there are no nested functions in rhs's, $p, p' \in P(t_j)$ implies $p \| p'$.

Definition 2.11 *Given a language* E *and a position* p*, we define* $R_p^{\triangleleft}(E)$ *as follows*

$$R_p^{*\triangleleft}(E) = E \cup \{t' \mid \exists t \in E, t \rightarrow_{[p, rhs's]}^+ t' \text{ by leftmost rewriting}\}$$

Example 2.12 $R = \{f(x) \rightarrow s(x),\ g(x,y) \rightarrow c(h(x), f(y)),\ h(x) \rightarrow f(x)\}$
$R_1^{*\triangleleft}(f(g(a,b))) = E \cup f(c(h(a), f(b))) \cup f(c(f(a), f(b))) \cup f(c(s(a), f(b))) \cup f(c(s(a), s(b)))$

Definition 2.13 *Let* $IRR_p(R) = \{s \in T_{C \cup F} \mid p \in Pos(s)$ *and* $s|_p$ *is irreducible*$\}$.

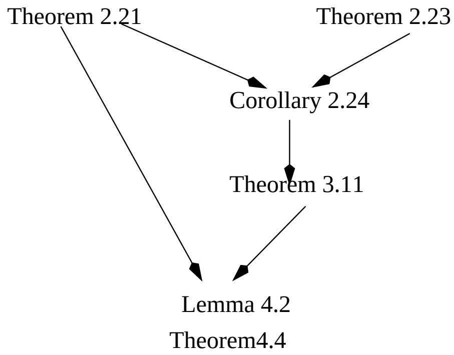
Theorem 2.14 *Let* t *be a term, and* $p \in \overline{Pos}(t)$. $IRR_p(R)$ *is a regular language and is recognized by an automaton that discriminates every position* $p' \in \overline{Pos}(t)$ *s.t.* $p' \not> p$.

Proof 2.15 *See [2].*
Let $\mathcal{A}_{irr}$ *an automaton that recognized* $IRR_\epsilon(R)$*. In the following, we will call* q^{irr} *the accepted state of* $\mathcal{A}_{irr}$.

2.5 Commutation of Rewriting and Left-Basic Derivation

Now, we need to show that the normal forms obtained without strategy are the same as these obtained with a leftmost strategy. For this, we have to show that a derivation can always be transformed to a left-basic derivation and if we have a left-basic derivation until a normal form then this derivation is leftmost.

The following figure shows the links between lemmas and theorems. Theorem 4.4 corresponds to the final result.



By the two following Lemmas, we see how to commute a derivation (in two steps, next in several steps).

Lemma 2.16 *Let R be a linear TRS and $s,\ t,\ u$ be terms. If*

$$s \rightarrow_{[p,g\rightarrow d]} t \rightarrow_{[q,l\rightarrow r]} u \qquad (1)$$

is a derivation in two steps s.t. q admit an antecedant in s denoted by p_0. Then, (1) can be commuted into:

$$s \rightarrow_{[p_0,l\rightarrow r]} t' \rightarrow_{[p,g\rightarrow d]} u' \qquad (2)$$

Now, let us suppose restrictions 5 and 4. Then, if (1) is leftmost, (2) is leftmost.
Remark : *R being linear, $g \rightarrow d$ is linear and so d is linear, consequently $u' = u$: The last term of derivation is preserved by commutation. Moreover, in (2), p does not have an antecedant in s.*

Lemma 2.17 *Let R be a linear TRS. If*

$$t_0 \rightarrow_{[p_0,l_0\rightarrow r_0]} t_1 \rightarrow \ldots t_{n-1} \rightarrow_{[pn-1,l_{n-1}\rightarrow r_{n-1}]} t_n \rightarrow_{[p_n,l_n\rightarrow r_n]} t_{n+1} \qquad (1)$$

is a derivation s.t. p_n admit an antecedant q_{n-1} in t_0. Then, (1) can be commuted in:

$$t_0 \rightarrow_{[q_{n-1},l_n\rightarrow r_n]} t'_1 \rightarrow_{[p_0,l_0\rightarrow r_0]} \ldots t'_n \rightarrow_{[p_{n-1},l_{n-1}\rightarrow r_{n-1}]} t'_{n+1} \qquad (2)$$

with $t'_{n+1} = t_{n+1}$ and p_0 has no antecedant in t_0.
Now, let us suppose restrictions 5 and 4. Then, if (1) is leftmost, (2) is leftmost too.

Now, let us define the notion of left-basic derivation.

Definition 2.18 *Let the following derivation:*

$$t_0 \rightarrow_{[p_0,l_0\rightarrow r_0]} t_1 \rightarrow \ldots t_n \rightarrow_{[p_n,l_n\rightarrow r_n]} t_{n+1} \qquad (1)$$

This derivation is said left-basic if: There exist sets of positions $B(t_0),\ B(t_1),\ldots,B(t_n)$ for the terms $t_0, t_1, \ldots, t_n$ s.t.

- $B(t_0) = PosF(t_0)$
- $\forall j,\ p_j \in B(t_j)$
- $\forall j,\ B(t_{j+1}) = \{p' \mid p' \leq p_j\} \cup \{p_j.v \mid v \in PosF(r_j)\} \cup \{p' \mid p_j \triangleleft p'\}$

Note that leftmost-innermost implies left-basic. The converse is false in the general case.

Lemma 2.19 *Let the following derivation:*

$$t_0 \rightarrow_{[p_0,l_0\rightarrow r_0]} t_1 \rightarrow \ldots t_n \rightarrow_{[p_n,l_n\rightarrow r_n]} t_{n+1} \qquad (1)$$

then, (1) is left-basic if and only if $\forall i \in\ \{1,\ldots,n\},\ p_i$ has no antecedent in t_{i-1}.

Lemma 2.20 *Let the following left-basic derivation:*

$$t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow \ldots t_n \qquad (1)$$

followed by the no-left-basic step:

$$t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1}$$

Let us remark that p_n admit an antecedent in t_{n-1}. Let $j \leq n$ the smaller integer s.t. p_n admit an antecedent in t_j.
Then, the following derivation obtained by commutation:

$$t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow \ldots t_j \rightarrow_{[q, l_n \rightarrow r_n]} t'_{j+1} \rightarrow_{[p_j, l_j \rightarrow r_j]} t'_{j+2} \rightarrow \ldots t'_n \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}]} t_{n+1} \qquad (2)$$

is left-basic.

Theorem 2.21 *Let R be a linear TRS. If $t_0 \rightarrow^* t_n$ (1) then, $t_0 \rightarrow^* t_n$ (2) by a left-basic derivation. Now, let us suppose restrictions 5 and 4. Then, if (1) is leftmost, (2) is leftmost too.*

Lemma 2.22 *Let R a TRS satisfying restrictions 1, 2, 3, 5. Let the following left-basic derivation:*

$$t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow \ldots t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1} \qquad (1)$$

and let $j \in \{0, \ldots, n\}$. If $\exists\ p' \in \overline{Pos}(t_j) - B(t_j)$ s.t. $t_j(p') \in \mathcal{F}$, then $\forall k > j$, $\exists p'_k \in \overline{Pos}(t_k) - B(t_k)$ s.t. $t_k|_{p'_k} = t_j|_{p'}$.

Theorem 2.23 *Let R a TRS satisfying restrictions 1, 2, 3, 5. Let the following left-basic derivation:*

$$t_0 \rightarrow^* t_{n+1} \text{ s.t. } t_{n+1} = t_o \downarrow \qquad (1)$$

Then, (1) is leftmost.

Corollary 2.24 *Let R a TRS satisfying restrictions 1 to 6. The normal forms obtained by a leftmost rewrite strategy are the same as those obtained without rewrite strategy.*

Proof 2.25 *Obviously, it is a consequence of Theorem 2.21 and of Theorem 2.23*

2.6 Starting Automaton

Let us define the initial automaton, i.e. the automaton that recognizes the data-instances of a given linear term t.

Definition 2.26 *We define the automaton $\mathcal{A}_{data}$ that recognizes the set of data-terms T(C) :*
$\mathcal{A}_{data} = (C, Q_{data}, Q_{data_f}, \Delta_{data})$ *where* $Q_{data} = Q_{data_f} = \{q_{data}\}$ *and* $\Delta_{data} = \{c(q_{data}, \ldots, q_{data}) \rightarrow q_{data} \mid c \in C\}$.

Given a linear term t, we define the automaton $\mathcal{A}_{t\theta}$ that recognizes the data-instances of t : $\mathcal{A}_{t\theta} = (C \cup F, Q_{t\theta}, Q_{t\theta_f}, \Delta_{t\theta})$ where

$$Q_{t\theta} = \{q^p \mid p \in \overline{Pos}(t)\} \cup \{q_{data}\}$$
$$Q_{t\theta_f} = \{q^\epsilon\} \ (q_{data} \text{ if } t \text{ is a variable})$$
$$\Delta_{t\theta} = \begin{array}{l} \left\{ t(p)(s_1, \ldots, s_n) \rightarrow q^p \mid p \in \overline{Pos}(t),\ s_i = \left| \begin{array}{l} q_{data} \text{ if } t|_{p.i} \text{ is a variable} \\ q^{p.i} \text{ otherwise} \end{array} \right. \right\} \\ \cup \Delta_{data} \end{array}$$

Note that $\mathcal{A}_{t\theta}$ discriminates each position p $\in \overline{Pos}(t)$ into q^p. On the other hand, $\mathcal{A}_{t\theta}$ is not deterministic, as soon as there is $p \in \overline{Pos}(t)$ s.t. $t|_p$ is a constructor-term. Indeed for any data-instance $t|_p\theta$, $t|_p\theta \rightarrow^*_{[\Delta_{t\theta}]} q^p$ and $t|_p\theta \rightarrow^*_{[\Delta_{t\theta}]} q_{data}$.

3 Recognizing $R_p^{*\lhd}(E)$

Notions introduced in this section are illustrated by example 3.12.

It may occur that the matches used in rewrite steps instantiate the variable by languages no recognized into states of $\mathcal{A}_E$. I.e the instances are not always (sub)terms of E. Let us see the following example:

Example 3.1 *Let E be the data-instances of $t = g(z)$ and*
$R = \{g(x) \stackrel{r1}{\to} h(b),\ h(y) \stackrel{r2}{\to} s(y)\}$.
$\mathcal{A}_{t\theta}$ *can be summarized by writing :* $\overset{q^\epsilon}{g}\ (\overset{q_{data}}{s^*(a)})$. *The rewrite steps issued from E are* $g(s^*(a)) \to_{[\epsilon, r1, x/s^*(a)]} h(b) \to_{[\epsilon, r2, y/b]} s(b)$. *Unfortunately,* $Q_{t\theta} = \{q^\epsilon,\ q_{data}\}$ *and the language recognized into q^ϵ (resp. q_{data}) is $g(s^*(a))$ (resp. $s^*(a)$).Thus, we do not have any states that can recognize $\{b\}$. This comes from the fact that $\{b\}$ is provided by the rhs $r1$. Therefore, we need to encode $\{b\}$ by additional states.*
So, we give the following definition.

Definition 3.2 *The non-variable arguments of functions in rhs's are encoded by the set of states Q_{arg} and the set of transitions Δ_{arg} as defined below:*

$$Q_{arg} = \{q^{i,p} \mid\ l_i \to r_i \in R,\ p \in Arg(r_i)\}$$
$$\Delta_{arg} = \{r_i(p)(q^{i,p.1}, \ldots, q^{i,p.n}) \to q^{i,p} \mid\ q^{i,p} \in Q_{arg}\}$$

where $Arg(r_i)$ are the non-variable argument positions in r_i, i.e.

$$Arg(r_i) = \{p \in \overline{Pos}(r_i) \mid\ \exists p_{fct} \in PosF(r_i),\ p > p_{fct}\}$$

Now, we define how to encode a version of each instantiated rhs.
Let $\mathcal{A} = (C \cup F, Q, Q_f, \Delta)$ an automaton that discriminates the position p into the state q, s.t. $Q \cap Q_{arg} = \emptyset$ (notation used in the continuation of this section). Let $Q' = Q \cup Q_{arg}$. We use states of the form d^p_σ where σ is a Q'-substitution, because rhs's may contain variables.

Definition 3.3 *The rhs's of rewrite rules are encoded by the sets of states Q_{arg} and*

$$\begin{aligned} D = \{d^{i,p}_\sigma \mid\quad & l_i \to r_i \in R,\ p \in Pos(r_i)\backslash Arg(r_i),\\ & \sigma \text{ is a } Q'\text{-substitution s.t. } dom(\sigma) = Var(r_i|_p)\}\end{aligned}$$

and the set of transitions

$$\begin{aligned}\Delta_d = \quad & \{r_i(p)(X_1, \ldots, X_n) \to d^{i,p}_{\sigma_1 \cup \ldots \cup \sigma_n} \mid\ l_i \to r_i \in R,\\ & \qquad p \in Pos(r_i)\backslash Arg(r_i),\ r_i(p) \in C\\ & \qquad \forall j,\ \sigma_j \text{ is any } Q'\text{-substitution s.t. } dom(\sigma_j) = Var(r_i|_{p.j}),\\ & \qquad \text{where } \forall j,\ X_j = \left| \begin{array}{l} x\sigma_j \mid\ r_i(p.j) \text{ is any variable } x\\ d^{i,p.j}_{\sigma_j} \text{ otherwise}\end{array}\right. \}\\ \cup\quad & \{r_i(p)(X_1, \ldots, X_n) \to d^{i,p}_\sigma \mid\ l_i \to r_i \in R,\ p \in PosF(r_i),\\ & \qquad \sigma \text{ is any } Q'\text{-substitution s.t. } dom(\sigma) = Var(r_i|_p)\\ & \qquad \text{where } \forall j,\ X_j = \left| \begin{array}{l} x\sigma \mid\ r_i(p.j) \text{ is any variable } x\\ q^{i,p.j} \in Q_{arg} \text{ otherwise}\end{array}\right. \}\\ \cup\quad & \{x\sigma \to d^{i,\epsilon}_\sigma \mid l_i \to r_i \in R,\ r_i \text{ is any variable } x,\\ & \qquad \sigma \text{ is any } Q'\text{-substitution s.t. } dom(\sigma) = \{x\}\}\end{aligned}$$

Thus, $r_i\sigma$ and only it, is recognized into the state $d^{i,\epsilon}_\sigma$.

Definition 3.4 *Let D^{sat}andΔ^{sat}_d the set of states and transitions obtained as in Definition 3.3 by replacing each state $d^{i,p}_\sigma \in D$ by $d^{i,p}_{sat,\sigma}$.*

The goal of these two similar encodings of rhs's is to recognized the instances of rhs's thanks to (D, Δ_d), and theirs descendants thanks to $(D^{sat}, \Delta^{sat}_d)$ and Δ^{sat} generated by the saturation process defined below:

Definition 3.5 *(*saturation*)*
Let Δ^{sat} be the set of transitions added in the following way: whenever there are $l_i \to r_i \in R$, a $(Q \cup Q_{arg})$-substitution σ s.t. $dom(\sigma) = Var(l_i) \cup Var(r_i)$ and $l_i\sigma \to^*_{\Delta_{t\theta} \cup \Delta_{arg} \cup \Delta_d^{sat}} q'$ where $q' \in \{q\} \cup D^{sat}$, add the transition $d^{i,\epsilon}_{sat,\sigma|_{Var(r_i)}} \to q'$.

Notation : $\Delta_d^{sat*} = \Delta_d^{sat} \cup \Delta^{sat}$.

Remark : Let us note $\mathcal{B}' = (\mathcal{C} \cup \mathcal{F}, Q \cup Q_{arg} \cup D^{sat}, \{q\}, \Delta \cup \Delta_{arg} \cup \Delta_d^{sat*})$. Then, $L(\mathcal{B}') = R^*_\epsilon(L(\mathcal{A}|_p))$ i.e the same as $R^{*\lhd}_\epsilon(L(\mathcal{A}|_p))$ except that the rewrite steps are not necessarily leftmost (see [1] for more details and explanations, here only *sat* differs).

We create another rhs's encoding. So, it permit us to have descendants of instances of rhs's obtained by a leftmost strategy. For example, consider the rhs $c(f(x), g(y))$. We check that instances of $f(x)$ are reduced to their normal forms, by any strategy (thanks to Corollary 2.24), before reducing instances of $g(y)$ by a leftmost strategy.

Definition 3.6 *For this, we consider the set of states:*

$$D_{spec} = D \cup D^{sat} \cup D^{sat} \times Q_{irr}$$

and the follow set of transitions where ($d_{\dots}^{\dots} q^{irr}$ denotes the pair $(d_{\dots}^{\dots}, q^{irr})$):

$$\begin{array}{rl}
\Delta_{spec} = & \bigcup_{l_i \to r_i \in R} \bigcup_{p \in Pos(r_i)/(PosF(r_i) \cup Arg(r_i))} \bigcup_{k \in \{1,\dots,ar(r_i(p))\}} \\
& \{r_i(p)(X_1, \dots, X_n) \to d^{i,p}_{sat,\sigma_1 \cup \dots \cup \sigma_n} \mid \forall j,\ \sigma_j \text{ is any } Q'\text{-substitution} \\
& \qquad\qquad \text{s.t. } dom(\sigma_j) = Var(r_i|_{p.j}), \\
& \qquad \text{where } \forall j,\ X_j = \left| \begin{array}{l} x\sigma_j \mid\ r_i(p.j) \text{ is any variable } x \\ d^{i,p.j}_{sat,\sigma_j} \text{ if } j = k \\ d^{i,p.j}_{sat,\sigma_j} q^{irr} \text{ if } j < k \\ d^{i,p.j}_{\sigma_j} \text{ otherwise} \end{array} \right. \quad \} \\
\cup & \{r_i(p)(X_1, \dots, X_n) \to d^{i,p}_{sat,\sigma} \mid\ l_i \to r_i \in R,\ p \in PosF(r_i), \\
& \qquad \sigma \text{ is any } Q'\text{-substitution s.t. } dom(\sigma) = Var(r_i|_p) \\
& \qquad \text{where } \forall j,\ X_j = \left| \begin{array}{l} x\sigma \mid\ r_i(p.j) \text{ is any variable } x \\ q^{i,p.j} \in Q_{arg} \text{ otherwise} \end{array} \right. \quad \} \\
\cup & \{x\sigma \to d^{i,\epsilon}_{sat,\sigma} \mid l_i \to r_i \in R,\ r_i \text{ is any variable } x, \\
& \qquad \sigma \text{ is any } Q'\text{-substitution s.t. } dom(\sigma) = \{x\}\}
\end{array}$$

Thus, $r_i\sigma$ is also recognized into the state $d^{i,\epsilon}_{sat,\sigma}$.

Now, we define an automaton that recognizes $R^{*\lhd}_p(L(\mathcal{A}))$.

Definition 3.7 *We define $\mathcal{B}^\lhd$ an automaton s.t.:*

$$\mathcal{B}^\lhd = (\mathcal{C} \cup \mathcal{F}, Q^\lhd, Q^\lhd_f, \Delta^\lhd)$$

where $Q^\lhd = D_{spec} \cup Q_{arg}$ and $Q_f = \{q\}$ and $\Delta^\lhd = \Delta_d \cup \Delta_d^{sat*} \otimes \Delta_{irr} \cup \Delta_{spec} \cup \Delta^{sat} \cup \Delta_{arg}$.
$\Delta_d^{sat*} \otimes \Delta_{irr}$ is obtained by running the automaton intersection algorithm on transition sets Δ_d^{sat*} and Δ_{irr}. Thus it encodes normal-forms of instances of rhs's.

Lemma 3.8 $L(\mathcal{B}^\lhd) = R^{*\lhd}_\epsilon(L(\mathcal{A}|_p))$.

Proof 3.9 *It comes from Corollary 2.24 and [1].*

Corollary 3.10 $L(\mathcal{A}[\mathcal{B}^\lhd]_p) = R^{*\lhd}_p(L(\mathcal{A}))$.

Theorem 3.11 *Let R be a rewrite system satisfying restrictions 1 to 6, and E be the set of data-instances of a given linear term t. If E is recognized by an automaton that discriminates position p into the state q, and possibly p' into q' for some $p' \in \overline{Pos}(t)$ s.t. $p' \not\geq p$, and some states q', then so is $R_p^{*\triangleleft}(E)$.*

Example 3.12 *Let $R = \{f(x) \rightarrow^1 s(x),\ h(x,y) \rightarrow^2 c(f(x), g(y)),\ g(x) \rightarrow^3 s(x)\}$ and $t = h(x,y)$. We consider only instances of t by constructors a, s, i.e. $E = h(s^*(a), s^*(a))$.*
We are looking for an automaton that recognized $R_\epsilon^{*\triangleleft}(E)$. For sake of simplicity we denote by σ any substitution. The only leftmost derivation is: $E \rightarrow_{[\epsilon,2,\sigma]} c(f(s^*(a)), g(s^*(a))) \rightarrow_{[1,1,\sigma]} c(s(s^*(a)), g(s^*(a))) \rightarrow_{[2,3,\sigma]} c(s(s^*(a)), s(s^*(a)))$. In the following, we give only sets of transitions.
$\Delta_{t\theta} = \{a \rightarrow q_{data},\ s(q_{data}) \rightarrow q_{data},\ h(q_{data}, q_{data}) \rightarrow q^\epsilon\}$ where final state is q^ϵ.
Let us define $\mathcal{B}^\triangleleft$.
$\Delta_d = \{c(d_\sigma^{2,1}, d_\sigma^{2,2}) \rightarrow d_\sigma^{2,\epsilon},\ f(q_{data}) \rightarrow d_\sigma^{2,1},\ g(q_{data}) \rightarrow d_\sigma^{2,2},\ s(q_{data}) \rightarrow d_\sigma^{1,\epsilon},\ s(q_{data}) \rightarrow d_\sigma^{3,\epsilon}\}$.
$\Delta_{spec} = \{c(d_{sat,\sigma}^{2,1}, d_\sigma^{2,2}) \rightarrow d_{sat,\sigma}^{2,\epsilon}, c(d_{sat,\sigma}^{2,1} q^{irr}, d_{sat,\sigma}^{2,2}) \rightarrow d_{sat,\sigma}^{2,\epsilon},\ f(q_{data}) \rightarrow d_{sat,\sigma}^{2,1},\ g(q_{data}) \rightarrow d_{sat,\sigma}^{2,2},\ s(q_{data}) \rightarrow d_{sat,\sigma}^{1,\epsilon},\ s(q_{data}) \rightarrow d_{sat,\sigma}^{3,\epsilon}\}$.
$\Delta_d^{sat} = \{c(d_{sat,\sigma}^{2,1}, d_{sat,\sigma}^{2,2}) \rightarrow d_{sat,\sigma}^{2,\epsilon},\ f(q_{data}) \rightarrow d_{sat,\sigma}^{2,1},\ g(q_{data}) \rightarrow d_{sat,\sigma}^{2,2},\ s(q_{data}) \rightarrow d_{sat,\sigma}^{1,\epsilon}$,
$s(q_{data}) \rightarrow d_{sat,\sigma}^{3,\epsilon}\}$.
$\Delta^{sat} = \{d_{sat,\sigma}^{2,\epsilon} \rightarrow q^\epsilon,\ d_{sat,\sigma}^{1,\epsilon} \rightarrow d_{sat,\sigma}^{2,1},\ d_{sat,\sigma}^{3,\epsilon} \rightarrow d_{sat,\sigma}^{2,2}\}$.
For the following set, we give only what we will use: $\Delta_d^{sat*} \otimes \Delta_{irr} \supseteq \{s(q_{data}) \rightarrow d_{sat,\sigma}^{3,\epsilon} q^{irr},\ s(q_{data}) \rightarrow d_{sat,\sigma}^{1,\epsilon} q^{irr},\ d_{sat,\sigma}^{1,\epsilon} q^{irr} \rightarrow d_{sat,\sigma}^{2,1} q^{irr}\}$.
Leftmost descendants are recognized indeed.

4 An automaton that recognizes $R_{left}^*(E)$

We look for an automaton that recognizes the sets of descendants according to a leftmost strategy.

Definition 4.1
$R_{left,p}^*(L) = \{s' \mid \exists s \in L,\ s \rightarrow_{[u_1,\ldots,u_n]}^* s' \text{ by a leftmost strategy with } \forall i\ u_i \geq p\}$.

Lemma 4.2 *Let R be a rewrite system satisfying restrictions 1 to 6 and E be the data-instances of a given linear term t.*
Let $p \in PosF(t)$, and L be a language s.t. $L|_p = E|_p$, and that is recognized by an automaton that discriminates every position $p' \in PosF(t) \mid p' \geq p$. Then,

$$R_{left,p}^*(L) = R_p^{*\triangleleft}(L) \quad if\ Succ(p) = \emptyset$$

Otherwise, let $Succ(p) = \{p_1, \ldots, p_n\}$ with $p_1 \triangleleft \ldots \triangleleft p_n$

$$R_{left,p}^*(L) = \left| \begin{array}{l} R_p^{*\triangleleft}[R_{left,p_1}^*(L)] \\ \cup\ R_p^{*\triangleleft}[R_{left,p_2}^*((R_{left,p_1}^*(L) \cap IRR_{p_1}(R)))] \\ \cup \ldots \cup\ R_p^{*\triangleleft}[R_{left,p_n}^*(\ldots(R_{left,p_1}^*(L) \cap IRR_{p_1}(R)) \ldots \cap IRR_{p_{n-1}}(R))] \end{array} \right.$$

and $R_{left,p}^*(L)$ is recognized by an automaton $\mathcal{A}'$ s.t. if $p' \in \overline{Pos}(t)$, $p' \not> p$ and $\mathcal{A}$ discriminates p' into q', then $\mathcal{A}'$ also discriminates p' into q'.

Proof 4.3 *See appendix B.*

Theorem 4.4 *Let R be a rewrite system satisfying restrictions 1 to 6, and E be the data-instances of a given linear term t.*

$$R_{left}^*(E) = R_{left,\epsilon}^*(E) \quad if\ t(\epsilon) \in F$$

Otherwise, let $Succ(p) = \{p_1, \ldots, p_n\}$ *with* $p_1 \lhd \ldots \lhd p_n$

$$R^*_{left}(E) = \left| \begin{array}{l} R^*_{left,p_1}(E) \\ \cup\ R^*_{left,p_2}((R^*_{left,p_1}(L) \cap IRR_{p_1}(R))) \\ \cup \ldots \cup R^*_{left,p_n}(\ldots(R^*_{left,p_1}(L) \cap IRR_{p_1}(R)) \ldots \cap IRR_{p_{n-1}}(R)) \end{array} \right.$$

and $R^*_{left}(E)$ *is effectively recognized by an automaton.*

Proof 4.5 *See appendix B.*

Example 4.6 *Let* $R = \{f(x) \to s(x),\ h(x,y) \to c(f(x), g(y)),\ g(x) \to s(x)\}$ *and* $E = h(g(s^*(a)), f(s^*(a)))$. *For clarity, we denote* $s^*(a)$ *by* $*$.
$R^*_{left}(E) = R^*_{left,\epsilon}(E) = R^{*\lhd}_{\epsilon}(R^*_{left,1}(E)) \cup R^{*\lhd}_{\epsilon}(R^*_{left,2}(R^*_{left,1}(E) \cap IRR_1(R))$.

- $R^*_{left,1}(E) \ = R^{*\lhd}_1(E)$
  $= E \cup h(s(*), f(*))$ *denoted by* L_1.

- $R^*_{left,2}(R^*_{left,1}(E) \cap IRR_1(R)) \ = R^*_{left,2}(h(s(*), f(*))$
  $= h(s(*), f(*) \cup h(s(*), s(*)$ *denoted by* L_2.

Finally, we obtain leftmost descendants which are:
$L1 \cup L2 \cup c(f(g(*)), g(f(*))) \cup c(s(g(*)), g(f(*))) \cup c(s(s(*)), g(f(*))) \cup c(s(s(*)), s(f(*))) \cup c(s(s(*)), s(s(*))) \cup c(f(s(*)), g(s(*))) \cup c(s(s(*)), g(s(*))) \cup c(f(s(*)), g(f(*)))$.

5 Conclusion

When computing descendants, taking some strategies into account is possible, keeping the same class of tree language (the regular ones) and assuming the same restrictions as in [2] for innermost strategy, plus three others (restrictions 4, 5, 6). So, the following question arises: are restrictions 4, 5, 6 really necessary to get regular sets of leftmost descendants?

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Appendix

A Proofs on Rewriting Commutation and Left-basic Derivation

A.1 Proof of Lemma 2.16

The fact that p has not antecedent it's obvious because of Definition 2.8. Either p occurs on the right of p_0, either p occurs on top of p_0.
Now, let us show that (1) $leftmot \Rightarrow$ (2) $leftmost$.
Let (1) leftmost. Let us classify $Var(r)$ from left to right. if $|Var(r)| = n$ then $Var(r) = \{x_1, \ldots, x_n\}$. Since (1) is leftmost, let us denote by $x_i = x$ the first variable instantiated by a term containing a defined-function. Then, $\forall j \in [1..i[$, $\sigma(x_j)$ do not contain function.
Let us suppose that p_0 are not a leftmost position in s. This is possible only if $\exists$ a function that occurs on the left of x in l instantiated by a term containing a function. Now it happens that it is not possible because of prohibition of permutative rules (see restriction 4) and because of variable preserving TRS (see restriction 5). Then, if we classify $Var(l)$ from left to right, we obtain the same order as $Var(r)$. Hence, (2) is leftmost.

A.2 Proof of Lemma 2.17

This proof is based on Lemma 2.16.
Let $ant(p_n, t_{n-1}) = q_0 \ \Rightarrow t_{n-1} \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}]} t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1}$ commute itself into $t_{n-1} \rightarrow_{[q_0, l_n \rightarrow r_n]} t'_n \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}]} t_{n+1}$. And p_{n-1} has not antecedent in t_{n-1}, and it is leftmost.
Let $ant(q_0, t_{n-2}) = q_1 \ \Rightarrow t_{n-2} \rightarrow_{[p_{n-2}, l_{n-2} \rightarrow r_{n-2}]} t_{n-1} \rightarrow_{[q_0, l_n \rightarrow r_n]} t'_n$ commute itself into $t_{n-2} \rightarrow_{[q_1, l_n \rightarrow r_n]} t'_n \rightarrow_{[p_{n-2}, l_{n-2} \rightarrow r_{n-2}]} t'_n$. And p_{n-2} has not antecedent in t_{n-2}, and it is leftmost.
...
By induction, let $ant(q_{n-2}, t_0) = q_{n-1} \ \Rightarrow t_0 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t_1 \rightarrow_{[q_{n-2}, l_n \rightarrow r_n]} t'_2$ commute itself into $t_0 \rightarrow_{[q_{n-1}, l_n \rightarrow r_n]} t'_1 \rightarrow_{[p_0, l_0 \rightarrow r_0]} t'_2$. And p_0 has not antecedant in t_0 and it is leftmost.
Hence, we obtain (2).

A.3 Proof of Lemma 2.19

Let (1) a left-basic derivation.
Let us suppose that $\exists i \in \{1 \ldots n\}$ s.t. p_i has an antecedent in t_{i-1}. By Definition 2.18, $p_i \in B(t_i)$ where $\forall j$, $B(t_i) = \{p' \mid p' \leq p_{i-1}\} \cup \{p_{i-1}.v \mid v \in PosF(r_{i-1})\} \cup \{p' \mid p_{i-1} \triangleleft p'\}$. Let $ant(p_i, t_{i-1}) = q$.
By Definition 2.8,

1. $q \in Pos(t_{i-1})$
2. - $p_i \triangleleft p_{i-1}$ *and* $p_i = q$ or, (A)
   - $\exists p' \in PosVar(r_{i-1})$ *with* $r_{i-1}|_{p'} = x$ *s.t.* $p_i = p_{i-1}.p'.w$ *and* $q = p_{i-1}.p''.w$ *where* p'' is an occurrence of variable x in l_{i-1}. (B)

It happens that (B) is impossible because of Definition 2.18; p_i would occur in forbidden position. (A) is impossible too, $not(p_i \triangleleft p_{i-1})$ else the derivation is not left-basic.

Hence, $\forall i \in \{1 \dots n\}$, p_i has not an antecedent in t_{i-1}.

Let $\forall i \in \{1 \dots n\}$, p_i has not an antecedent in t_{i-1}.
$not(p_i \lhd p_{i-1})$ else we find an antecedent, and $p_i \neq p_{i-1}.p'.w$ where $p' \in PosVar(r_{i-1})$ $(r_{i-1}|_{p'} = x)$ because else we find $q = p_{i-1}.p''.w$ where p'' is an occurrence of x in p_{i-1}. Hence, it is left-basic.

A.4 Proof of Lemma 2.20

Let $ant(p_n, t_{n-1}) = q_0$,. Derivation can be commuted and we obtain:
$t_0 \to t_1 \to \dots t_j \to t_{n-1} \to_{[q_0]} t'_n \to_{[p_{n-1}]} t_{n+1}$ where p_{n-1} has not antecedent in t_{n-1} according to Lemma 2.16.
By induction, let $j < n$ the smaller integer s.t. p_n admit an antecedent in t_j. Let denote by q this antecedent. By Lemma 2.17, we can commute and we obtain:
$t_0 \to \dots t_j \to_{[q, l_n \to r_n]} t'_{j+1} \to_{[p_j]} \dots \to t'_n \to_{[p_{n-1}]} t_{n+1}$ (4) and p_j has not antecedent in t_j.
q has not an antecedent in t_{j-1} since according to the remark $j < n$ is the smaller integer s.t. p_n admit an antecedent in t_j. Moreover, since (1) is left-basic, $t_0 \to \dots t_j$ is left-basic. And since $\forall i \in \{j+1 \dots n-1\}$ p_i has not antecedant in t_i for the derivation $t_j \to_{[q]} t'_{j+1} \to_{[p_j]} \dots \to t'_n \to_{[p_{n-1}]} t_{n+1}$ (3) then (3) is left-basic according to Lemma 2.19. And, since q has not antecedent in t_{j-1} then (4) is left-basic.

A.5 Proof of Lemma 2.21

Let $i \in \{1 \dots n\}$ s.t. $t_0 \to \dots t_i$ left-basic derivation and $t_0 \to \dots t_i \to t_{i+1}$ (1') no left-basic.
By running of Lemma 2.20, (1') can be commuted in $t_0 \to \dots t_j \to_{[q]} t'_{j+1} \to \dots t'_i \to t_{i+1}$ (2') with $ant(p_i, t_j) = q$ and (2') is left-basic. And by Lemma 2.17, (2') is leftmost if (1') is leftmost.
We use Lemma 2.20, many times are necessary, and so $t_0 \to^* t_n$ by a left-basic derivation. We proceed in the same way with Lemma 2.17, and (2) is leftmost if (1) is leftmost.

A.6 Proof of Lemma 2.22

Let $t_0 \to_{[p_0, l_0 \to r_0]} t_1 \to \dots t_n \to_{[p_n, l_n \to r_n]} t_{n+1}$ (1) a left-basic derivation, and let $j \in \{0, \dots, n\}$.
Let $p' \in \overline{Pos}(t_j) - B(t_j)$ s.t. $t_j(p') \in \mathcal{F}$. Let us take $k = j+1$. By absurd, let us suppose that $\neg(\exists p'_k \in \overline{Pos}(t_k) - B(t_k)$ s.t. $t_k|_{p'_k} = t_J|_{p'})$. I.e. $\forall p'_k \in \overline{Pos}(t_k) - B(t_k)$ s.t. $t_k|_{p'_k} \neq t_J|_{p'}$. Then, we have to remove $t_j|_{p'}$ during the rewriting step.

- if $p' \lhd p_j$ then this is impossible.
- if p' below p_j, $p' \in PosVar(l_j)$, seeing the constructor-based system, then the rewrite rule $l_j \to r_j$ should have to eliminate variables. It happens that we have a restrictions removing this model of rules.

By induction, we obtain the result for $k = j+2, \dots, n$.

A.7 Proof of Theorem 2.23

By absurd, let us suppose that $\exists j, p'$ s.t. $t_j \to_{[p']} u$ with $p' \lhd p_j$. According to Lemma 2.22, $\nexists k > j$ s.t. $p_k = p'$. Then, $p' \notin B(t_{j+1})$ and $t_{j+1} \in \mathcal{F}$. According to Lemma 2.22, $\exists p'_{n+1} \in Pos(t_{n+1})$ s.t. $t_{n+1}|_{p'_{n+1}} = t_j|p'$. That is in opposite to the fact as t_{n+1} be normalized.

B Proofs on Computation of Leftmost Descendants

B.1 Proof of Lemma 4.2

- If $Succ(p) = \emptyset$, then $\forall s \in L, \forall p' \in Pos(s), (p' > p \implies s(p') \in C)$. p is the only one function position of $s|_p$. In rhs's, it may have several function position that are incomparable (in opposition with nested). $R_p^{*\triangleleft}(L)$ compute leftmostly. Therefore, $R_p^{*\triangleleft}(L) = R^*_{left,p}(L)$.
  We get $\mathcal{A}'$ by Theorem 3.11.

- Let $Succ(p) = \{p_1, \ldots, p_n\}$ with $p_1 \triangleleft \ldots \triangleleft p_n$. Let $s \in L. \forall\, i \in \{1, \ldots, n\}$, s can be rewritten leftmostly at position p_i, but descendants at position p_{i-1} must have computed and normalized at this position before, since $p_{i-1} \triangleleft p_i$. Let us note L_1 set of terms obtained from s after leftmost rewriting at position p_1,..., L_n set of terms obtained from s after leftmost rewriting at position p_n.
  We have:
  $L_1 = R^*_{left,p_1}(L)$
  $L_2 = R^*_{left,p_2}(R^*_{left,p_1}(L) \cap IRR_{p_1}(R))$
  $\ldots$
  $L_n = R^*_{left,p_n}(L_{n-1} \cap IRR_{p_{n-1}}(R))$.
  Remark : Those descendants are obtained by left-basic rewriting.
  $L_1, \ldots, L_n$ can be possibly rewritten at position p. Moreover, since there is not variable occuring on the left of a function (and not under another) in rhs's (by restriction 6), so, $R_p^*(L_1) \cup \ldots \cup R_p^*(L_n) = R^*_{left,p}(L)$.
  L is recognized by an automaton $\mathcal{A}$ that discriminates every position $p' \in PosF(t)$ s.t. $p' \geq p$ and so, since $\forall i \in \{1, \ldots, n\}$, $p_i > p$, every position p' s.t. $p' \geq p_i$.
  By induction hypothesis, L_1 is recognized by an automaton $\mathcal{A}_1$ that discriminates p and every position $p' \in PosF(t)$s.t. $p' \not> p_1$ and so, in particular, every position $p' \geq p_i$ $\forall i \in \{2, \ldots, n\}$ and every position $p' \not> p$. By Theorem 3.11, $R_p^{*\triangleleft}(L_1)$ is recognized by an automaton that discriminates positions $p' \not> p$.
  By Theorem 2.14, $IRR_{p_i}(R)$ is recognized by an automaton that discriminates every position $p' \in PosF(t)$ s.t. $p' \not> p_i$ ($p' \not> p_1$ for $IRR_{p_1}(R), \ldots, p' \not> p_n$ for $IRR_{p_n}(R)$). For $j \in \{1, \ldots, n\}$, let us suppose that L_{j-1} is recognized by an automaton that discriminates every position $p' \not> p_{j-1}$ (i.e. positions that are discriminated before computing $R^*_{left,p_{j-1}}(\ldots)$ minus positions that are below p_{j-1}). By Lemma 2.5, $L_{j-1} \cap IRR_{p_{j-1}}(R)$ is recognized by an automaton that discriminates every position $p' \not> p_{j-1}$, so in particular $p' \geq p_j$. L_j is recognized by an automaton that discriminates every position $p' \not> p_j$ (i.e. positions that are discriminated before computing $R^*_{left,p_j}(\ldots)$ minus positions that are below p_j) and in particular every position $p' \not> p$. By Lemma 3.11, $R_p^{*\triangleleft}(L_j)$ is recognized by an automaton that discriminates every position $p' \not> p$.
  Finally, by union, we obtain an automaton that discriminates p and preserves discrimination of positions $p' \not> p$.

B.2 Proof of Theorem 4.4

We have two cases:

- If $\epsilon \in PosF(t)$, obviously $R^*_{left}(E) = R^*_{left,\epsilon}(E)$.

- If $\epsilon \notin PosF(t)$, and $Succ(p) = \{p_1, \ldots, p_n\}$ $with$ $p_1 \triangleleft \ldots \triangleleft p_n$, $\forall\, i, j$ s.t. $p_i \triangleleft p_j$, leftmost descendants at position p_j can be computed after have normalized those at position p_i. Then obviously,$R^*_{left}(E) = R^*left, p_1(E) \cup \ldots$

$\cup \; R^{*}left, p_{n-1}(\ldots(R^{*}left, p_1(L) \cap IRR_1(R)))\ldots) \cup \ldots$
$\cup \, R^{*}left, p_n(\ldots(R^{*}left, p_1(L) \cap IRR_1(R)))\ldots)$