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Rapport de Recherche

Faithful embedding on finite orders classes

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Faithful embedding on finite orders classes

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Abstract

We investigate, in the particular case of finite orders classes, the notion of faithful embedding among relations introduced in 1971 by R. Fraïssé. This notion leads to the study, by forbidden substructures, of orders relations in a way which isn't the usual one and thus point out specific structural characteristics. In this paper we show that, for most of the known classes of orders, any order belonging to such a class admits a faithful embedding also belonging to this class.

Key words: embedding, extension, faithful embedding, finite orders, isomorphism, partially ordered set, poset.

1 Introduction

We start by fixing some notations and definitions on orders, for further informations and particularly for orders classes that we mention but that we don't study, we refer to the books of B.S.W. Schröder [5] and of W.T. Trotter [6]. Also, notations and definitions proper to particular sections will be introduced as necessary.

An order P is an ordered pair $(V(P), <_P)$, where $V(P)$ is a (finite) set and $<_P$ is an irreflexive, transitive (and thus antisymmetric) binary relation on $V(P)$. For the sequel consider that $x, y \in V(P)$ and that $X \subseteq V(P)$. The notation $x \leq_P y$ signifies either $x <_P y$ or $x = y$, and the notation $x {-\!\!<}_P y$ signifies that both $x <_P y$ and $\forall z \in V(P)$ if $x \leq_P z <_P y$ then $x = z$. When these two elements are distinct, they are *comparable* (resp. *incomparable*), denoted by $x \sim_P y$ (resp. $x \parallel_P y$), if either $x <_P y$ or $y <_P x$ (resp. neither $x <_P y$ nor $y <_P x$). The element x is *maximal*, respectively *minimal*, if for every $z \in V(P)$ we have that $x \leq_P z$, respectively $z \leq_P x$, implies $x = z$. The

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element x is *isolated* if it is both a maximal and a minimal element. The set X is a *chain* (resp. an *antichain*) of P when every two distinct elements of X are comparable (resp. incomparable). The *rank* of x in P, denoted $\text{rank}_P(x)$, is the maximal cardinal, minus one, of a chain of P having x for greatest element. For every $k \in \mathbb{N}$, all the elements of P with rank k is denoted by $\mathcal{R}_P(k)$, that is $\mathcal{R}_P(k) = \{x : x \in V(P) : \text{rank}_P(x) = k\}$. The *height* of P, denoted by $h(P)$, is the maximal cardinal, minus one, of a chain of P. With each subset X of $V(P)$ is associated the *suborder* $(X, (X \times X) \cap <_P)$ of P induced by X, denoted by $P[X]$. When $X = V(P) - \{x\}$ we rather use the notation $P - x$. Consider two orders P and Q. An *isomorphism* from P onto Q is a one-to-one mapping $f : V(P) \longrightarrow V(Q)$ such that for all $x, y \in V(P)$, $x <_P y$ if and only if $f(x) <_Q f(y)$. The order Q is an *extension* of P whenever $P = Q[V(P)]$. This extension is *sharp* if moreover $|V(Q)| - |V(P)| = 1$. The order P *embeds* into the order Q if there exists a suborder of Q isomorphic to P. Thus, if Q is an extension of P then P embeds into Q, but the converse is false.

The notion of faithful embedding, which have for particular interest to allow to increase the semantic of logic's systems while preserving syntactic rules, has first been introduced for arbitrary relations in 1971 by R. Fraïssé [2]. This notion has then been expressed for finite orders, in the following sense, in 1993 by R. Fraïssé and W. Hazim-Sharif [3]:

Definition 1 *An order P admits a faithful embedding relatively to an order Q if (i) P doesn't embed into Q, and (ii) Q admits a sharp extension in which P doesn't embed.*

The general problem of determining whether a couple of orders fulfills this notion is quite obvious. Indeed, as indicated in [2], it is sufficient to notice that: if an order P doesn't embed into an order Q, then for any $x \notin V(Q)$ either P doesn't embed into $(V(Q) \cup \{x\}, <_Q)$ or P doesn't embed into $(V(Q) \cup \{x\}, <_Q \cup \{(x, y) : y \in V(Q)\})$. Consequently, the notion of faithful embedding becomes interesting only when it is restricted to subclasses of finite orders. Under these considerations, following an idea of R. Bonnet, R. Fraïssé and W. Hazim-Sharif [3] investigated classes of fixed height orders. In their study, they introduced the notions of h-inextensibility and of h-extensibility, which admit the following generalizations to an arbitrary class of orders $\mathcal{C}$:

Definition 2 *An order P is* $\mathcal{C}$-inextensible *for an order Q whenever (i) P, Q belong to $\mathcal{C}$, (ii) P doesn't embed into Q, and (iii) P embeds into any sharp extension of Q belonging to $\mathcal{C}$.*

Definition 3 *An order P is* $\mathcal{C}$-extensible *whenever (i) P belongs to $\mathcal{C}$, and (ii) every order Q in $\mathcal{C}$, in which P doesn't embed, admits a sharp extension, belonging to $\mathcal{C}$, in which P still doesn't embed.*

Let $\mathcal{H}_k$ be the height k orders class, that is the class of orders whose maximal cardinality chains contain $k+1$ elements. Let $\mathcal{I}so(P)$ be the set of isolated elements of an order P. An element of an order P, is *max dominating*, respectively *min dominating*, if it is greater, respectively smaller, in P than every element in $V(P)$ being not maximal, respectively not minimal. The results obtained by R. Fraïssé and W. Hazim-Sharif [3] can then be expressed as: there exists $\mathcal{H}_1$-extensible orders, there exists $\mathcal{H}_1$-inextensible orders, and, for $k \geq 2$, any order P such that either $\mathcal{I}so(P)$ is empty, or such that $P - \mathcal{I}so(P)$ has neither a max dominating nor a min dominating element, is $\mathcal{H}_k$-extensible.

In addition to its links with logic's systems, this notion of faithful embeddings has as interest to lead to the study, by forbidden substructures, of the orders relations in a way which isn't usual. Consequently, specific characteristics of known orders classes are enlightened and particular structural aspects have to be used. Thus, in this paper we extend the study of faithful embeddings to various classes of orders and we show that for most of the known classes the extensibility criterion is fulfilled. In Section 2 we recall some known or easy facts, on embeddings and on orders' extensibility, that we need further. In Section 3 we show, for several orders classes, that only two kinds of extensions are sufficient to insure that these classes fulfill the extensibility criterion. Among them we find the class of interval orders, of k-dimensional orders, of semi-orders, of weak-orders, of connected orders, of lattices, and of in-tree orders. The pairs of extensions are the parallels and series extensions, the Min-Max substitution and series extensions, the universal complement and series extensions, and the minimal pendant and series extensions. In Section 4 we show, using an unbounded number of extensions, that orders in $\mathcal{C}$ are $\mathcal{C}$-extensible whenever $\mathcal{C}$ is either the class of disconnected orders, or the class of width k orders, or the class of indecomposable orders. In Section 5 we study the class $\mathcal{H}_k$, of height k orders, and we show that any height k order having one rank without a maximal element is $\mathcal{H}_k$-extensible.

We now end by fixing some others notations and definitions that we use all over in the sequel. First of all, given any strictly positive integer k, the set $\{1, \cdots, k\}$ will be denoted by $[k]$. Given an order P, let $x, y \in V(P)$ and let $X \subseteq V(P)$. The *predecessors* set of x is the set $D_P(x) = \{y : y \in V(P) : y <_P x\}$, $D_P[x] = D_P(x) \cup \{x\}$ and the set of *immediate predecessors* of x is the set $D^{im}_P(x) = \{y : y \in V(P) : y \prec_P x)\}$. The same holds for the *successors* sets with $U_P(x)$, $U_P[x]$ and $U^{im}_P(x)$. The set $\mathrm{Max}_P(X)$ (resp. $\mathrm{Min}_P(X)$) denotes the set of maximal (resp. minimal) elements of $P[X]$. The *dual* order of P is the order P^d where $V(P^d) = V(P)$ and $<_{P^d} = \{(x, y) : x, y \in V(P) : (y, x) \in <_P\}$. Given a family $(P_i)_{i \in I}$ of pairwise disjoint orders, that is for $i, j \in I$ we have that $V(P_i) \cap V(P_j) = \emptyset$ whenever $i \neq j$, the *parallel composition* of the P_i's is the order $P = \bigoplus_{i \in I} P_i$ where $V(P) = \bigcup_{i \in I} V(P_j)$ and $<_P = \bigcup_{i \in I} <_{P_i}$. When $|I| = 2$, with for example $I = \{1, 2\}$, we rather use $P_1 \oplus P_2$. Given P_1 and P_2, two disjoint orders, the *left series composition* of P_2 by P_1, or equivalently, the *right series composition* of P_1 by P_2, is the order $P = P_1 \otimes P_2$ where $V(P) = V(P_1) \cup V(P_2)$ and $<_P = <_{P_1} \cup <_{P_2} \cup \{(x_1, x_2) : x_1 \in V(P_1), x_2 \in V(P_2)\}$. When $P_1 = (\{x\}, \emptyset)$, respectively $P_2 = (\{y\}, \emptyset)$, we rather use $x \otimes P_2$, respectively

$P_1 \otimes y$. Given a mapping ψ from a set A into a set B, to simplify notations, we also use ψ to denote the corresponding mapping from the subsets of A into the subsets of B. That is, for $X \in 2^A$ and for $Y \in 2^B$, we have $\psi(X) = \bigcup_{x\in X} \psi(x)$ and $\psi^-(Y) = \bigcup_{y\in Y} \psi^-(y)$, where $\psi^-(y) = \{x : x \in X : \psi(x) = y\}$. Also, when ψ is injective (one-to-one), for every $b \in B$, whenever $\psi^-(b)$ exists, we use indifferently $\psi^-(y)$ either for the singleton or for the element itself.

2 Preliminaries

2.1 Embedding

We recall some known facts about embeddings and we provide some proofs for the reader's convenience.

Fact 1 *Let P and Q be two orders such that P embeds into Q by ψ. Then, for every $x \in V(Q)$, such that $\psi^-(x)$ exists, we have that $P[D_P(\psi^-(x))]$, respectively $P[U_P(\psi^-(x))]$, embeds into $Q[D_Q(x)]$, respectively $Q[U_Q(x)]$, by ψ restricted to $D_P(\psi^-(x))$, respectively $U_P(\psi^-(x))$.*

From this fact immediately follows the following property:

Property 1 *Let P and Q be two orders such that P embeds into Q by ψ. Let $x, y \in V(Q)$ such that both $\psi^-(x)$ and $\psi^-(y)$ exist, then we have:*

(i) $x \prec_Q y$ *implies that* $\psi^-(x) \prec_P \psi^-(y)$.

(ii) $|\psi^-(D^{im}_Q(x))| = |D^{im}_Q(x)|$ *implies that* $D^{im}_P(\psi^-(x)) = \{\psi^-(z) : z \in D^{im}_Q(x)\}$, *dually* $|\psi^-(U^{im}_Q(x))| = |U^{im}_Q(x)|$ *implies that* $U^{im}_P(\psi^-(x)) = \{\psi^-(z) : z \in U^{im}_Q(x)\}$.

(iii) $x \in Max(Q)$, *respectively* $x \in Min(Q)$, *implies that* $\psi^-(x) \in Max(P)$, *respectively* $\psi^-(x) \in Min(P)$.

(iv) if x is the least, respectively greatest, element of Q, then $\psi^-(x)$ is the least, respectively greatest, element of P.

Proof. Case (i): first notice that for any order H we have that $x \prec_H y$ if and only if both $x <_H y$ and $U_H(x) \cap D_H(y) = \emptyset$. Then, this case immediately follows from Fact 1 since for any mapping φ from U to V and for any subsets A, B of V, holds $\varphi^-(A) \cap \varphi^-(B) = \varphi^-(A \cap B)$.

Case (ii): from Case (i) it follows that $\{\psi^-(z) : z \in D^{im}_Q(x) : \psi^-(z) \in V(P)\} \subseteq D^{im}_P(\psi^-(x))$, and thus that $\{\psi^-(z) : z \in D^{im}_Q(x)\} \subseteq D^{im}_P(\psi^-(x))$, since $|\psi^-(D^{im}_Q(x))| = |D^{im}_Q(x)|$. Assume that there exists $t \in D^{im}_P(\psi^-(x)) - \{\psi^-(z) : z \in D^{im}_Q(x)\}$. Then we have $\psi(t) <_Q \psi(\psi^-(x)) = x$, and thus there exists $\alpha \in V(Q)$ such that $\psi(t) <_Q \alpha \prec_Q x$. Consequently we obtain $t = \psi^-(\psi(t)) <_P \psi^-(\alpha) \prec_P \psi^-(x)$: a contradiction. The dual case follows the same lines.

Case (iii): as for any $x \in Max(Q)$, respectively $x \in Min(Q)$, holds $U_Q(x) = \emptyset$, respectively $D_Q(x) = \emptyset$, then the result directly follows from Fact 1.

Case (iv): as for every $y \in V(Q) - \{x\}$ we have that $x <_Q y$, respectively $y <_Q x$, and as for every $z \in V(P) - \{\psi^-(x)\}$ we have that $\psi(z) \in V(Q) - \{x\}$, this case is also an immediate consequence of Fact 1. $\square$

2.2 Extensibility

Let Q and P be two orders such that $V(P) \cap V(Q) = \{x\}$. We define the order Q_x^P by (i) $V(Q_x^P) = V(Q) \cup V(P)$, and (ii) $\forall\ a \neq b \in V(Q_x^P)$ we have $a <_{Q_x^P} b$ whenever either $\{a, b\} \subseteq V(P)$ and $a <_P b$ or $\{a, b\} \subseteq V(Q)$ and $a <_Q b$ or $a \in V(P)$, $b \in (V(Q) - V(P))$ and $x <_Q b$ or $b \in V(P)$, $a \in (V(Q) - V(P))$ and $a <_Q x$.

Fact 2 *Let P and Q be two orders such that $V(P) \cap V(Q) = \{x\}$. Then Q_x^P is a sharp extension of Q whenever $|V(P)| = 2$.*

Lemma 1 *Let P and Q be two orders such that P does not embed into Q. Let $x \notin V(Q)$, let $y \in V(Q)$ and let Z be any order such that $V(Z) = \{x, y\}$. If P embeds, by ψ, into Q_y^Z then both $\psi^-(x)$ and $\psi^-(y)$ exist.*

Proof. As P does not embed into Q then $\psi^-(x)$ exists. Now, assume that $\psi^-(y)$ does not exist. Thus, P embeds into $Q_y^Z - y$, which is in contradiction with the fact that $Q_y^Z - y$ is isomorphic to Q. $\square$

By taking y as the greatest element of Q we obtain the following corollary.

Corollary 1 *Let P and Q be two orders such that P does not embed into Q. Assume that Q has a greatest element denoted by $\top$ and let $x \notin V(Q)$. If P embeds into $Q \otimes x$ by ψ, then P has a greatest element with an unique immediate predecessor.*

Proof. First of all, notice that $Q \otimes x = Q_\top^Z$ with $Z = (\{\top, x\}, (\top, x))$. Now, by Lemma 1, we have that both $\psi^-(x)$ and $\psi^-(\top)$ exist. From Property 1 (iii) it follows that $\psi^-(x)$ is the greatest element of P. Now, as $D^{im}_{Q\otimes x}(x) = \{\top\}$, by Property 1 (ii) we have that $D^{im}_P((\psi^-(x))) = \{\psi^-(\top)\}$. $\square$

3 Some easy classes

In this section, in order to show that the classes we investigate fulfill the extensibility criterion, we only need to exhibit two sharp extensions per class.

3.1 Parallels and series extensions

Directly from the proof that any order is $\mathcal{O}$-extensible, when $\mathcal{O}$ is the class of all orders, we can express the following lemma that we call Fraïssé's lemma:

Lemma 2 [Fraïssé] *Let $\mathcal{C}$ be an order class stable by the parallel composition with an element and stable by the left or right series composition by an element. Then any order in $\mathcal{C}$, on at least two elements, is $\mathcal{C}$-extensible.*

Proof. Let P, Q be two orders belonging to $\mathcal{C}$ such that P does not embed into Q. Let $x \notin V(Q)$. First, assume that P embeds into $Q \oplus x$ by ψ. Then, as P does not embed into Q, we have that $\psi^{-}(x)$ exists and thus, by Fact 1, we have that $\psi^{-}(x)$ is an isolated element of P. Second, assume, for example, that P embeds into $Q \otimes x$ by ϕ. Again, as P does not embed into Q, we have that $\phi^{-}(x)$ exists and thus, by Property 1 (iv), we obtain that P has a greatest element: which is in contradiction with $1 <_{\mathbb{N}} |V(P)|$. The case P embeds into $x \otimes Q$ follows exactly the same lines. □

As an immediate consequence we obtain that the classes of N-free orders (see B.S.W. Schröder [5] page 50), series-parallel orders (see B.S.W. Schröder [5] page 220), threshold orders, truncated lattices (see B.S.W. Schröder [5] page 137), forest of in-tree orders (or CS-forest see W.T. Trotter [6] page 116), k-dimensional orders with $k \geq 2$ (see B.S.W. Schröder [5] page 169 or W.T. Trotter [6] page 9), interval orders (see B.S.W. Schröder [5] page 186 or W.T. Trotter [6] page 86), cycle-free orders (see W.T. Trotter [6] page 123), and decomposable orders (see B.S.W. Schröder [5] page 203 or W.T. Trotter [6] page 24) are $\mathcal{C}$-extensible when $\mathcal{C}$ denotes the class under consideration.

Extending the idea of operators underlying the Lemma 2, we can obtain the extensibility criterion for three more classes.

3.2 Min-Max substitution and series extensions

Lemma 3 *Let $\mathcal{C}$ be an order class stable by the right, respectively the left, series composition by an element, and stable by the substitution of any of its maximal, respectively minimal, elements by a two elements antichain. Then any order in $\mathcal{C}$ is $\mathcal{C}$-extensible.*

Proof. Let P, Q be two orders belonging to $\mathcal{C}$ such that P does not embed into Q. Let $x \notin V(Q)$. First, assume that P embeds into $Q \otimes x$ by ϕ. Then, as P does not embed into Q, we have that $\phi^{-}(x)$ exists and thus, by Property 1 (iv), we obtain that P has a greatest element. Second, let $y \in Max(Q)$, let $Z = (\{y, x\}, \emptyset)$ and assume that P embeds into Q_y^Z by ψ. Then, by Lemma 1, $\psi^{-}(x)$ and $\psi^{-}(y)$ exist and moreover they belong, by Property 1 (iii), to $Max(P)$: a contradiction.

The case P embeds both into $x \otimes Q$ and into Q_y^Z, for $y \in Min(Q)$, follows exactly the same lines. □

As an immediate consequence we obtain that weak orders (or weakly ordered set see B.S.W. Schröder [5] page 181), semi-orders (see B.S.W. Schröder [5] page 197 or W.T. Trotter [6] page 192), and connected orders on at least two elements (see B.S.W. Schröder [5] page 44 or W.T. Trotter [6] page 4) are $\mathcal{C}$-extensible when $\mathcal{C}$ denotes the class to which they belong.

3.3 Universal complement and series extensions

Following G. Birkhoff [1] (cf. page 6), a *lattice* is an order L any two of whose elements have a greatest lower bound or "meet" denoted by $x \wedge y$, and a least upper bound or "join" denoted by $x \vee y$.

For any order P, we denote $\top_P$, respectively $\bot_P$, the greatest, respectively the least, element of P when it exists. For any order P, on at least two elements, having both a greatest and a least element, and for any $x \notin V(P)$, we denote by $P \rhd x$ the order $(V(P) \cup \{x\}, <_{_P} \cup \{(x, \top), (\bot, x)\})$. Obviously, for any lattice Q, on at least two elements, and for any element $x \notin V(Q)$, the order $Q \rhd x$ is still a lattice.

Proposition 1 *Let $\mathcal{C}$ be the class of lattices, then, any order P in $\mathcal{C}$, with $3 <_{_\mathbb{N}} |V(P)|$, is $\mathcal{C}$-extensible.*

Proof. Let P and Q two lattices such that P does not embed into Q, $3 <_{_\mathbb{N}} |V(P)|$ and $x \notin V(Q)$. First assume that P embeds into $Q \otimes x$, which is obviously a lattice, thus, by Corollary 1, we obtain that $|D_{_P}^{im}(\top_P)| = 1$. Second, assume that P embeds into $Q \rhd x$ by ψ. Then, as P does not embed into Q, we have that $\psi^-(x)$ exists. Notice that both $D_{_{Q \rhd x}}(x) = \{\bot_{Q \rhd x}\}$ and $U_{_{Q \rhd x}}(x) = \{\top_{Q \rhd x}\}$, then, by Fact 1, since P is a lattice on at least three elements, we obtain that there exists both $\psi^-(\top_{Q \rhd x})$ and $\psi^-(\bot_{Q \rhd x})$. By Property 1 (iv) we have that $\psi^-(\top_{Q \rhd x}) = \top_P$ and $\psi^-(\bot_{Q \rhd x}) = \bot_P$. Now, as $\bot_{Q \rhd x} \prec_{_{Q \rhd x}} x \prec_{_{Q \rhd x}} \top_{Q \rhd x}$, Property 1 (i) implies that $\bot_P \prec_{_P} \psi^-(x) \prec_{_P} \top_P$. Finally, the fact that $\top_P$ has an unique immediate predecessor, implies that P is isomorphic to the 3 elements chain: which contradicts that $3 <_{_\mathbb{N}} |V(P)|$. $\square$

3.4 Minimal pendant and series extensions

Recall that an order P is an *in-tree order* if it has a greatest element and for every $x \in V(P)$ we have that $P[U_{_P}[x]]$ is a total order. That is, an in-tree order is any order whose transitive reduction is an anti-arborescence.

For an order P, having $\top_P$ has greatest element, and for an element $x \notin V(P)$, we denote by $P \circledwedge x$ the order $(V(P) \cup \{x\}, <_{_P} \cup \{(x, \top_P)\})$. Obviously, if P is an in-tree order then $P \circledwedge x$ still belongs to this class.

Proposition 2 *Let $\mathcal{C}$ be the class of in-tree orders, then, any order P in $\mathcal{C}$ with $|V(P)| > 2$ is $\mathcal{C}$-extensible.*

Proof. Let P be an in-tree order on more than 2 elements, let Q be an in-tree order in which P does not embed, and let $x \notin V(Q)$. First, assume that P embeds into $Q \otimes x$, which obviously is still an in-tree order, thus, by Corollary 1, we obtain that $|D^{im}_{P}(\top_P)| = 1$. Second, assume that P embeds into $Q \circledcirc x$ by ψ. Then, as P does not embed into Q, we have that $\psi^-(x)$ exists. But, as $U_{Q \circledcirc x}(x) = \{\top_{Q \circledcirc x}\}$ and as $1 <_{\mathbb{N}} |V(P)|$, Fact 1 and the fact that P is a in-tree order imply that $\psi^-(\top_{Q \circledcirc x})$ exists. Moreover, Property 1 (iv) insures that $\psi^-(\top_{Q \circledcirc x}) = \top_P$. Finally, as $x \prec_{Q \circledcirc x} \top_{Q \circledcirc x}$, Property 1 (i) implies that $\psi^-(x) \prec_P \top_P$. Consequently, as $\top_P$ has an unique immediate predecessor, we obtain that P is isomorphic to the 2 elements chain: which contradicts that $2 <_{\mathbb{N}} |V(P)|$. $\square$

4 Some more classes

For the orders classes that we investigate in this section, we need a variable number of sharp extensions to show that they fulfill the extensibility criterion.

4.1 Disconnected orders

Recall that a disconnected order is any order whose comparability graph is disconnected. Using at least as many sharp extensions as the number of connected components of the extended order we show that the class of disconnected orders fulfills the extensibility criterion.

Theorem 1 *Let $\mathcal{C}$ be the class of disconnected orders then any order in $\mathcal{C}$ is $\mathcal{C}$-extensible.*

Proof. Let P and Q be two disconnected orders. Then, we have that $P = \bigoplus_{i \in [n]} P_i$ with $2 \leq_{\mathbb{N}} n$, $Q = \bigoplus_{j \in [m]} Q_j$ with $2 \leq_{\mathbb{N}} m$, and that all the P_i's and the Q_i's are connected orders on at least one element. Assume that P does not embed into Q and let $x \notin V(Q)$. Let $S_0 = (V(Q) \cup \{x\}, <_Q)$, and for $j \in [m]$ let $S_j = (V(Q) \cup \{x\}, <_Q \cup \{(x, y) : y \in V(Q) - V(Q_j)\})$. Notice that all the S_i's are obviously sharp extensions of Q, and that, moreover, for every $j \in [m]$ the order $S_j[V(Q) - V(Q_j)]$ has x for least element. We claim that there exists $i \in \{0\} \cup [m]$ such that P still does not embed into S_i. On the contrary assume that, for every $i \in \{0\} \cup [m]$, P embeds into S_i by ϕ_i. First notice that since P does not embed into Q then, for every $i \in \{0\} \cup [m]$, we have that $\phi_i^-(x) \in V(P)$. Consequently, when $i \in [m]$, there exists an unique $j \in [n]$ such that $\phi(V(P_j)) \cap (V(Q) - V(Q_i)) \neq \emptyset$. In the sequel we denote by j_i such a j. Consider S_m, then up to a bijection we can assume that $j_m = n$. First assume that there exists $i \in [m-1]$ such that $j_i \in [n-1]$. Thus, since $\bigoplus_{j \in [n-1]} P_j$ embeds into Q_m by the appropriate restriction of ϕ_m, and since P_n embeds into Q_i by the appropriate restriction of ϕ_i, we immediately obtain an embedding of P into $Q_m \oplus Q_i$: a contradiction. Second we have that for every $i \in [m]$, $j_i = m$ and thus for every $i \in [m]$ we have that $\bigoplus_{j \in [n-1]} P_i$ embeds into Q_i by the appropriate restriction of ϕ_i. Now, consider S_0, then either there exists $\alpha \in [m]$ such that P_n embeds into Q_α, by the appropriate restriction of ϕ_0, and thus, for any $\beta \in [m]$ with $\beta \neq \alpha$, we immediately obtain an embedding of P

into $Q_\alpha \oplus Q_\beta$: a contradiction. Or, P_n embeds into $(\{x\}, \emptyset)$, by the appropriate restriction of ϕ_0, and thus P_n embeds into Q_i for every $i \in [m]$. Consequently for every $\alpha, \beta \in [m]$ with $\beta \neq \alpha$, we obtain an embedding of P into $Q_\alpha \oplus Q_\beta$: a contradiction. $\square$

4.2 Width k orders

Recall that the width of an order is the maximal cardinal of one of its antichains. Using at least as many sharp extensions as the height of the extended order, we show that the class of width k orders, fulfills the extensibility criterion for any $1 <_{\mathbb{N}} k$. Directly from this proof we obtain a similar result for the class of orders of width less or equal to k but greater than one.

Theorem 2 *If $\mathcal{C}$ is the class of width k orders, with $2 \leq_{\mathbb{N}} k$, then any order in $\mathcal{C}$ is $\mathcal{C}$-extensible.*

Proof. Let P and Q be two orders such that $w(P) = w(Q) = k$ and $2 \leq_{\mathbb{N}} k$. Assume that P does not embed into Q, let $x \notin V(Q)$ and, for every $y \in V(Q)$, let $Z_y = (\{y, x\}, (y, x))$. Notice that for every $y \in V(Q)$ we have that $w(Q_y^{Z_y}) = k$. To prove the theorem by contradiction we show that if P embeds into every sharp extension of Q then $w(P) = 1$. For that purpose we show, by induction, that for every $0 \leq_{\mathbb{N}} i \leq_{\mathbb{N}} h(P)$, $|\mathcal{R}_P(i)| = 1$. The fact that $|\mathcal{R}_P(0)| = 1$ is an immediate consequence of Property 1 (iv) when considering the sharp extension $x \otimes Q$. Consider $0 \leq_{\mathbb{N}} n <_{\mathbb{N}} h(P)$ and assume that for every $0 \leq_{\mathbb{N}} j \leq_{\mathbb{N}} n$ we have that $\mathcal{R}_P(j) = \{c_j\}$. Let $y \in \mathcal{R}_Q(n)$. Notice that such an y exists since as P embeds into every sharp extension of Q then holds $h(P) - 1 \leq_{\mathbb{N}} h(Q)$. Now assume that P embeds into $Q_y^{Z_y}$ by φ, and recall that, by Lemma 1, we have that both $\varphi^-(y)$ and $\varphi^-(x)$ exist. Now, since $y \in \mathcal{R}_{Q_y^{Z_y}}(n)$, it follows, by Fact 1, that $\varphi^-(y) \in \{c_0, \cdots, c_n\}$. First assume that $\varphi^-(y) = c_j$ with $j <_{\mathbb{N}} n$, then by Property 1 (i) it follows that $\varphi^-(x) = c_{j+1}$. Consequently, using Fact 1, we obtain that $P[U_P[c_{j+1}]]$ embeds into $Q_y^{Z_y}[U_{Q_y^{Z_y}}[x]]$. Then, since $Q_y^{Z_y}[U_{Q_y^{Z_y}}[x]]$ is clearly isomorphic to $Q[U_Q[y]]$, it follows that $P[U_P[c_{j+1}]]$ embeds into $Q[U_Q[y]]$. Then, as $P = P[\{c_0, \cdots, c_j\}] \otimes P[U_P[c_{j+1}]]$, the fact that, both $P[\{c_0, \cdots, c_j\}]$ is a chain and that $y \in \mathcal{R}_Q(n)$ with $j <_{\mathbb{N}} n$, immediately implies that P embeds into Q: a contradiction. Second we have that $\varphi^-(y) = c_n$. Then as an immediate consequence of Property 1 (i) and (ii), we obtain that $U_P^{im}(c_n) = \{\varphi^-(x)\}$, and thus it immediately follows that $\mathcal{R}_P(n+1) = \{\varphi^-(x)\}$. $\square$

4.3 Indecomposable orders

Given an order P, a subset X of $V(P)$ is an *interval*, or an *autonomous*, of P provided that for all $u, v \in X$ and $x \in V(P) - X$, $(u, x) \in <_P$ if and only if $(v, x) \in <_P$, and $(x, u) \in <_P$ if and only if $(x, v) \in <_P$. Clearly, $\emptyset$, $V(P)$ and $\{x\}$, where $x \in V(P)$, are intervals of P, called *trivial* intervals. Then , an order is *indecomposable*, or *prime*, if it has at least 3 elements and if all of its intervals are trivial. Notice that this implies that it has in fact at least 4 elements. For further details about these notions, due to T. Gallai, we refer to P. Ille [4].

For an order Q and for any $m \in Max(Q)$, respectively $m \in Min(Q)$, we denote by $Q^{x,m}$, respectively $Q_{x,m}$, the order $(V(Q) \cup \{x\}, <_Q \cup\{(y,x) : y \in V(Q) - \{m\}\})$, respectively $(V(Q) \cup \{x\}, <_Q \cup\{(x,y) : y \in V(Q) - \{m\}\})$.

Lemma 4 *Let Q be an indecomposable order and let $x \notin V(Q)$, then:*

(i) for every $m \in Max(Q)$, having a maximal ideal size, the order $Q' = Q^{x,m}$ is indecomposable,

(ii) for every $m \in Max(Q)$, such that for every $t \in D^{im}_Q(m) \cap Min(Q)$ we have $U_Q(t) \neq \{m\}$, the order $Q' = (V(Q) \cup \{x\}, <_Q \cup\{(x,m)\})$ is indecomposable,

(iii) for every $p \in Min(Q)$, such that $U_Q(p) = \{m\}$, the order $Q' = (V(Q) \cup \{x\}, <_Q \cup\{(p,x)\})$ is indecomposable.

Proof. We proceed by contradiction. Let X be a none trivial interval of Q', that is $X \subsetneq V(Q')$ and $2 \leq_{\mathbb{N}} |X|$. As $Q = Q'[V(Q)]$, and as Q is indecomposable, it then follows that both $x \in X$ and $|X| = 2$. Thus, assume that $X = \{x, y\}$.
For case (i): either $x \parallel_{Q^{x,m}} y$, then $y = m$, and thus Q has m for greatest element, which is impossible. Or $y <_{Q^{x,m}} x$, then $D_{Q^{x,m}}(y) = V(Q^{x,m}) - \{x, m, y\}$, and thus, since $D_{Q^{x,m}}(y) = D_Q(y)$ and $y \neq m$, it follows that $D_Q(y) = V(Q) - \{m, y\}$. But now, the ideal size assumption on m, implies either that m is the greatest element of Q, or that $D_Q(m) = D_Q(y)$: each of them being impossible.
For case (ii): either $x \parallel_{Q'} y$, then $y \in Min(Q')$ and $U_Q(y) = U_{Q'}(y) = U_{Q'}(x) = \{m\}$. But, since $Min(Q) = Min(Q')$, this contradicts the assumption on m. Or, $x <_{Q'} y$, then $y = m$, and then, it follows that $D_Q(m) = D_{Q'}(m) - \{x\} = D_{Q'}(x) - \{m\} = D_{Q'}(x) = \emptyset$. Thus m is an isolated element of Q, which contradicts that Q is indecomposable.
For case (iii): either $x \parallel_{Q'} y$, then $y \in Max(Q')$ and $D_Q(y) = D_{Q'}(y) = D_{Q'}(x) = \{p\}$, consequently $y = m$. Thus $D_Q(p) = U_Q(m) = D_Q(m) - \{p\} = U_Q(p) - \{q\} = \emptyset$, which implies that $\{m, p\}$ is a none trivial interval of Q and this is impossible. Or $y <_{Q'} x$, then $y = p$, and then, it follows that $U_Q(p) = U_{Q'}(p) - \{x\} = U_{Q'}(x) - \{p\} = U_{Q'}(x) = \emptyset$. Thus p is an isolated element of Q, which contradicts that Q is indecomposable. □

Lemma 5 *Let P be an indecomposable order, let Q be any order, and let $m \in Max(Q)$. For any $x \notin V(Q)$, if P does not embed into Q but embeds into $Q^{x,m}$, then $Max(P) = \{x_1, x_2\}$ and $D_P(x_1) = V(P) - Max(P)$.*

Proof. Assume that P embeds into $Q^{x,m}$ by ϕ. Then, as P does not embed into Q, we have that $\psi^-(x)$ exists. Consequently, from Fact 1, it follows that

$\forall y \in V(P) - \{\psi^-(x)\}$ we have $y <_P \psi^-(x)$ whenever $y \neq \psi^-(m)$. But as P is indecomposable, it follows that $\psi^-(m)$ exists, which gives the Lemma. $\square$

Lemma 6 *Let P be an indecomposable order on at least 6 elements. Let Q be any order, let $m_a \in Max(Q)$ and let $m_i \in Min(Q)$. For any $x \notin V(Q)$, if P does not embed into Q but embeds both into Q^{x,m_a} and into Q_{x,m_i}, then $|\{m : m \in Max(P) \cup Min(P) : |D_P(m)| + |U_P(m)| = 1\}| \leq_{\mathbb{N}} 1$.*

Proof. From Lemma 5, on P and Q, and on P^d and Q^d, it follows that $Max(P) = \{m_{a_1}, m_{a_2}\}$ with $D_P(m_{a_2}) \subsetneq D_P(m_{a_1}) = V(P) - Max(P)$, and that $Min(P) = \{m_{i_1}, m_{i_2}\}$ with $U_P(m_{i_2}) \subsetneq U_P(m_{i_1}) = V(P) - Min(P)$. Since P is indecomposable, we have that $Max(P) \cap Min(P) = \emptyset$, and thus that $|D_P(m_{a_1})| \geq 2$, $|U_P(m_{i_1})| \geq 2$, $|D_P(m_{a_2})| \geq 1$, and $|U_P(m_{i_2})| \geq 1$. By contradiction, we now assume that $|D_P(m_{a_2})| = |U_P(m_{i_2})| = 1$. Consequently, $\forall z \in V(P) - (Max(P) \cup Min(P))$ holds $m_{i_1} <_P z <_P m_{a_1}$, $z \parallel_P m_{i_2}$, and $z \parallel_P m_{a_2}$. Thus $V(P) - (Max(P) \cup Min(P))$ is an interval of P which becomes none trivial for $6 \leq_{\mathbb{N}} |V(P)|$. $\square$

Theorem 3 *If $\mathcal{C}$ is the class of indecomposable orders, then any order P in $\mathcal{C}$, with $6 \leq_{\mathbb{N}} |V(P)|$ is $\mathcal{C}$-extensible.*

Proof. Let P be an indecomposable order on at least 6 elements, let Q be any indecomposable order in which P does not embed, and let $x \notin V(Q)$. By contradiction assume that P embeds into every indecomposable extension of Q. Let $m_a \in Max(Q)$ and let $m_i \in Min(Q)$ having respectively a maximal ideal size and a maximal filter size. Since by Lemma 4 (i), on Q and on Q^d, we obtain that Q^{x,m_a} and Q_{x,m_i} are indecomposable extensions of Q, then Lemma 5, on both P and Q and on both P^d and Q^d, implies that $Max(P) = \{map_1, map_2\}$ with $D_P(map_1) = V(P) - Max(P)$, and $Min(P) = \{mip_1, mip_2\}$ with $U_P(mip_1) = V(P) - Min(P)$. Now assume that, there exists $q \in Min(Q)$ such that $U_Q(q) = \{mq\}$. Let $Q' = (V(Q) \cup \{x\}, <_Q \cup \{(q, x)\})$, then, by Lemma 4 (iii), Q' is an indecomposable extension of Q. Thus, assume that P embeds into Q' by ϕ. Then, as P does not embed into Q, $\phi^-(x)$ exists, and, by Property 1 (iii), it follows that $\phi^-(x) \in Max(P)$. Then, since P is connected, Fact 1 implies that $\phi^-(q)$ exists. But now, again from Fact 1 we have that $|U_P(\phi^-(q))| \leq_{\mathbb{N}} |U_{Q'}(q)| = 2$, from Property 1 (iii) we obtain that $\phi^-(q) \in Min(P)$, and from Property 1 (ii) it follows that $D_P^{im}(\phi^-(x)) = \{\phi^-(q)\}$. Consequently, we obtain that $\phi^-(q) = mip_1$, but that implies $|V(P)| \leq_{\mathbb{N}} 4$: a contradiction. If there exists $q \in Max(Q)$, such that $D_Q(q) = \{mq\}$, we obtain a similar contradiction with $Q' = (V(Q) \cup \{x\}, <_Q \cup \{(x, q)\})$. Consequently, for any $miq \in Min(Q)$ and for any $maq \in Max(Q)$, by Lemma 4 (ii), it follows that the orders $Q_{miq} = (V(Q) \cup \{x\}, <_Q \cup \{(miq, x)\})$ and $Q_{maq} = (V(Q) \cup \{x\}, <_Q \cup \{(x, maq)\})$ are indecomposable extensions of Q. Thus, on the one hand, P embeds into Q_{miq} by ψ and both we have $\psi^-(x) \in Max(P)$ and $|D_P(\psi^-(x))| + |U_P(\psi^-(x))| = 1$. On the other hand, P embeds into Q_{maq}

by ϕ, and we have both $\phi^-(x) \in Min(P)$ and $|D_P(\phi^-(x))| + |U_P(\phi^-(x))| = 1$. Which, all together, contradicts Lemma 6. $\square$

To complete the study of faithful embeddings on the class $\mathcal{C}$ of indecomposable orders, notice that, since any indecomposable order has a suborder isomorphic to the only -up to isomorphism- indecomposable order on 4 elements (for example $P = (\{1,3,2,4\}, \{(1,3),(2,4),(2,3)\})$), it is then sufficient to consider indecomposable orders on 5 elements. But then, following the proof of Theorem 3, we obtain that orders with more than two maximal or two minimal elements and orders with neither a minimal nor a maximal pendant (where x is, for example, a minimal pendant in an order Q whenever $x \in Min(Q)$ and $|U_Q(x)| = 1$) are $\mathcal{C}$-extensible. Consequently, it only remains to study the behavior of orders isomorphic to $P = (\{1,3,2,4,c\}, \{(1,3),(2,4),(2,3),(2,c),(c,3)\})$.

5 Conclusion

If for the class $\mathcal{H}_k$ we are not able to solve the general case, nevertheless we can strengthen a little the result of R. Fraïssé and W. Hazim-Sharif [3]. This only leaves open the case of orders in $\mathcal{H}_k$, with $1 <_{\mathbb{N}} k$, having a maximal element in each level of its rank decomposition.

We show that every order (of height k) having at least one level of its rank decomposition without a maximal element is $\mathcal{H}_k$-extensible.

Proposition 3 *Any order P in $\mathcal{H}_k$, such that $|\{i : i \in [h(P)+1] : \mathcal{R}_P(i-1) \cap Max(P) \neq \emptyset\}| \neq h(P)+1$, is $\mathcal{H}_k$-extensible.*

Proof. We proceed by contradiction. We assume that there exists Q in $\mathcal{H}_k$, such that P does not embed into Q but embeds into every sharp extension of Q belonging to $\mathcal{H}_k$. Let $x \notin V(Q)$, and, for $i \in \{0, \cdots, k\}$, let $Q_i = (V(Q) \cup \{x\}, <_Q \cup \{(y,x) : y \in \bigcup_{j=0}^{i-1} \mathcal{R}_Q(i)\})$. Notice that $Q_i \in \mathcal{H}_k$, and thus assume that P embeds into Q_i by ϕ_i. Since P does not embed into Q, for every $i \in \{0, \cdots, k\}$ we have that $\phi_i^-(x)$ exists, and, by Property 1 (iii), it follows that $\phi_i^-(x) \in Max(P)$. As, by Fact 1, for every $p \in V(P)$ we have that $\mathrm{rank}_P(p) \leq_{\mathbb{N}} \mathrm{rank}_Q(\phi(p))$, it then only remains to show that, for every $i \in \{0, \cdots, k\}$, we have that $i \leq_{\mathbb{N}} \mathrm{rank}_P(\phi_i^-(x))$. Let $c = (x_0, \cdots, x_k)$ be a maximal sized chain of P. Then, as $h(Q_i) = k$, $(\phi_i(x_0), \cdots, \phi_i(x_k))$ is also a maximal sized chain of Q_i and thus $\mathrm{rank}_Q(\phi(x_j)) = j$ for $j \in \{0, \cdots, k\}$. Consequently, $(x_0, \cdots, x_{i-1}, \phi_i^-(x))$ is a chain of P and thus $i \leq_{\mathbb{N}} \mathrm{rank}_P(\phi_i^-(x))$. $\square$

Two other classes, which seem interesting to study and which are not so easy, are the one of planar orders (see W.T. Trotter [6] page 66) and the one of tree-like orders (or poset tree orders see W.T. Trotter [6] page 117).

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