4 rue Léonard de Vinci
BP 6759
F-45067 Orléans Cedex 2
FRANCE
http://www.univ-orleans.fr/lifo

Rapport de Recherche

On Compatible Normal Odd Partitions in Cubic Graphs

J.L. Fouquet and J.M. Vanherpe
LIFO, Université d'Orléans

Rapport n^o **RR-2007-09**

On Compatible Normal Odd Partitions in Cubic Graphs

J.L. Fouquet and J.M. Vanherpe

L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans, 45067 Orléans Cedex 2, FR

Abstract

A *normal odd partition* $\mathcal{T}$ of the edges of a cubic graph is a partition into *trails* of odd length (no repeated edge) such that each vertex is the end vertex of exactly one trail of the partition and internal in another trail. For each vertex v, we can distinguish the edge $e_{\mathcal{T}}(v)$ for which this vertex is pending. Three normal odd partitions are compatible whenever these distinguished edges are distinct for each vertex. We study here this notion and show that a cubic 3-edge colourable graph can always be provided with three compatible normal odd partitions . The Petersen graph have this property and we can construct other cubic graphs with chromatic index 4 with the same property. At last, we give a connexion with the Fan Rapaud [3] conjecture and propose a new conjecture which, if true, would imply this conjecture.

Key words: Cubic graph; Edge-partition;

1 Introduction

For basic graph-theoretic terms, we refer the reader to Bondy and Murty [2]. Following Bondy [1], a *walk* in a graph G is sequence $W := v_0e_1v_1 \dots e_kv_k$, where $v_0, v_1, \dots, v_k$ are vertices of G, and $e_1, e_2 \dots, e_k$ are edges of G and v_{i-1} and v_i are the ends of e_i, $1 \leq i \leq k$. v_0 and v_k are the *end vertices* and e_1 and e_k are the *end edges* of this walk, while $v_1, \dots, v_{k-1}$ are the *internal vertices* and $e_2, \dots, e_{k-1}$ are the *internal edges*. The *length* $l(W)$ of W is the number of edges (namely k). W is *odd* whenever k is odd and *even* otherwise. The walk W is a *trail* if its edges $e_1, e_2, \dots, e_k$ are distincts and a *path* if its vertices $v_0, v_1, \dots, v_k$ are distincts. If $W := v_0e_1v_1 \dots e_kv_k$, is a walk of G $W' := v_ie_{i+1} \dots e_jv_j$ $(0 \leq i \leq j \leq k)$ is a *subwalk* of W (*subtrails* and *subpaths* are defined analogously). If v is an internal vertex of a walk W with ends x and y, $W(x, v)$ and $W(v, y)$ are the subwalks of W obtained in cutting W in v. Conversely if W_1 and W_2 have a common end v, the *concatenation* of these two walks *on* v gives rise to a new walk (denoted by $W_1 + W_2$) with v as an internal vertex. When no confusion, is possible, it will be convenient to omit the edges in the description of a walk, that is $W := v_0 e_1 v_1 \dots e_k v_k$ will be shortened in $W := v_0 v_1 \dots v_k$.

Let $G = (V, E)$ be a cubic graph (loops and multiple edges are allowed) and let $\mathcal{T} = \{T_1, T_2 \dots, T_k\}$ be a partition of $E(G)$ into trails.

Definition 1 A partition $\mathcal{T} = \{T_1, T_2 \dots, T_k\}$ of $E(G)$ into trails is *normal* when every vertex is the end vertex of one trail of $T_i \in \mathcal{T}$ and internal in $T_j \in \mathcal{T}$. The length of a normal partition is the maximum length of the trails in the partition, that is $\max\{l(T_i) | T_i \in \mathcal{T}\}$.

When $\mathcal{T} = \{T_1, T_2 \dots, T_k\}$ is a normal partition, we must have $k = \frac{n}{2}$. Moreover, we can associate to each vertex the unique edge with end v which is the end edge of a trail of $\mathcal{T}$. We shall denote this edge by $e_{\mathcal{T}}(v)$ and it will be convenient to say that $e_{\mathcal{T}}(v)$ is the *marked* edge associated to v. When it will be necessary to illustrate our purpose by a figure the marked edge associated to a vertex will be figurate by a $\vdash$ close to this vertex. Let v be a vertex, v is an internal vertex in $T_i \in \mathcal{T}$ and an end vertex in $T_j \in \mathcal{T}$ (as an end of $e_{\mathcal{T}}(v)$), we can associate to v the set $E_{\mathcal{T}}(v)$ containing the end vertices of T_i. It must be clear that T_i and T_j are not necessarily distinct, in this case we have $v \in E_{\mathcal{T}}(v)$. Let x and y be the ends of T_i, one of these two vertices (say x) is certainly different from v. Let $T'_i = T_i(x, v) + T_j$ and $T'_j = T_i(v, y)$. We can thus transform $\mathcal{T}$ in a new normal partition $\mathcal{T}'$ by an operation of *switching* defined by:

$$\mathcal{T}' = \mathcal{T} - \{T_i, T_j\} + \{T'_i, T'_j\}$$

Definition 2 A normal partition $\mathcal{T} = \{T_1, T_2 \dots, T_{\frac{n}{2}}\}$ of $E(G)$ into trails is *odd* when every trail in $\mathcal{T}$ is odd. For each trail of odd length $T_i \in \mathcal{T}$ let us say that an edge $e = uv$ of T_i is *odd* (resp. *even*) whenever the subtrails of T_i obtained in deleting e have odd (resp. even) lengths.

Given two normal partitions $\mathcal{T} = \{T_1, T_2 \dots, T_{\frac{n}{2}}\}$ and $\mathcal{T}' = \{T'_1, T'_2 \dots, T'_{\frac{n}{2}}\}$, $A_{\mathcal{T}\mathcal{T}'}$ is the set of vertices such that $e_{\mathcal{T}}(v) = e_{\mathcal{T}'}(v)$.

Definition 3 Two normal partitions $\mathcal{T}$ and $\mathcal{T}'$ of $E(G)$ into trails are *compatible* when $e_{\mathcal{T}}(v) \neq e_{\mathcal{T}'}(v)$ for every vertex $v \in V(G)$ (in other words $A_{\mathcal{T}\mathcal{T}'} = \emptyset$).

Given three normal partitions $\mathcal{T} = \{T_1, T_2 \dots, T_{\frac{n}{2}}\}$, $\mathcal{T}' = \{T'_1, T'_2 \dots, T'_{\frac{n}{2}}\}$ and $\mathcal{T}" = \{T"_1, T"_2 \dots, T"_{\frac{n}{2}}\}$, we let $A = A_{\mathcal{T}\mathcal{T}'} \cup A_{\mathcal{T}'\mathcal{T}"} \cup A_{\mathcal{T}'\mathcal{T}"}$. We shall say that G has three compatible normal odd partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}"$ whenever these partitions are pairwise compatible (in other words $A = \emptyset$).

In [4] we showed that a cubic graph without loop can always be provided with 3 compatible normal partitions.

Theorem 4 *[4] A cubic graph G has three compatible normal partitions if and only if G has no loop.*

It turns out that normal odd partitions are directly associated to perfect matchings and it is rather natural to ask whether the problem of finding three compatible normal odd partitions is connected to the problem of the edge colouring. Motivated by this perspective, we show that cubic graphs with chromatic index 3 can be provided with three compatible normal odd partitions. Moreover, there are cubic (non trivial) graphs with chromatic index 4 with that property (in particular the Petersen graph and the Flower Snarks).

2 Preliminary results

2.1 Switching equivalence

In [4] we have proved that any two normal partitions $\mathcal{T}$ and $\mathcal{T}'$ of a cubic graph are *switching equivalent*, that is, we can transform $\mathcal{T}$ in $\mathcal{T}'$ by a sequence of at most $2n$ switchings.

Assume that a normal odd partition $\mathcal{T}'$ is obtained from the normal odd partition $\mathcal{T}$ by an operation of switching, this operation of switching will be said an *odd switching*. If we can transform a normal odd partition $\mathcal{T}$ in $\mathcal{T}'$ by a sequence of odd switching operations, $\mathcal{T}$ and $\mathcal{T}'$ are said *odd switching equivalent*.

Theorem 5 *Any two normal odd partitions of a cubic graph G are odd switching equivalent.*

Proof Assume that we have chosen two normal odd partitions $\mathcal{T}$ and $\mathcal{T}'$ which are not odd switching equivalent and such that $A_{\mathcal{T}\mathcal{T}'}$ has maximum cardinality.

In particular, we have $A_{\mathcal{T}\mathcal{T}'} \neq V(G)$. Let $v \notin A_{\mathcal{T}\mathcal{T}'}$, u_1, u_2, and u_3 being the neighbors of v. For sake of simplicity e_1, e_2 and e_3 respectively denote the edges vu_1, vu_2 and vu_3. Assume without loss of generality that $e_{\mathcal{T}}(v) = e_2$ and $e_{\mathcal{T}'}(v) = e_3$. We denote T (resp. T') the trail of $\mathcal{T}$ (resp. $\mathcal{T}'$) which contains v as an internal vertex.

As a matter of fact the vertex v is an end point of T, more precisely

$$T = ve_2u_2 \dots u_1e_1ve_3u_3 \dots x$$

otherwise there would be a normal odd partition $\mathcal{T}''$ obtained from $\mathcal{T}$ in using an odd switching operation on v and such that $e_{\mathcal{T}''}(v) = e_3$, a contradiction since $A_{\mathcal{T}''\mathcal{T}'}$ would have a greater size than $A_{\mathcal{T}\mathcal{T}'}$.

Similarly $T' = ve_3u_3 \dots u_1e_1ve_2u_2 \dots x'$.

But now, in using odd switching operations on v we obtain normal odd partition $\mathcal{T}_1$ and $\mathcal{T}_1'$ respectively odd switching equivalent to $\mathcal{T}$ and $\mathcal{T}'$.

$$\mathcal{T}_1 = \mathcal{T} - \{T\} \cup \{ve_1u_1 \dots u_2e_2ve_3u_3 \dots x\},$$
$$\mathcal{T}_1' = \mathcal{T}' - \{T'\} \cup \{vve_1u_1 \dots u_3e_3ve_2u_2 \dots x'\}.$$

It follows that $|A_{\mathcal{T}_1\mathcal{T}_1'}| > |A_{\mathcal{T}\mathcal{T}'}|$, a contradiction. $\square$

Theorem 6 *Let G be a cubic graph then G has an odd normal partition if and only if G has a perfect matching.*

Proof Let M be a perfect matching in G. Then $G - M$ is a $2-$factor of G. let us give any orientation to this $2-$factor and for each vertex v let us denote the outgoing edge $o(v)$. For each edge $e = uv \in M$, let P_{uv} be the path of length 3 obtained in concatenating $o(u)$ uv and $o(v)$. Then $T = \{P_{uv} | uv \in M\}$ is a normal odd partition (of length 3) of G. Conversely let $\mathcal{T} = \{T_1, T_2 \dots, T_{\frac{n}{2}}\}$ be a normal odd partition of G. A vertex $v \in V(G)$ is internal in exactly one trail of $\mathcal{T}$. The edges of this trail being alternatively odd and even, v is incident to exactly one odd edge. Hence the odd edges so defined induce a perfect matching of G. $\square$

Given an odd normal partition $\mathcal{T}$ of G, we can define the *associated* perfect matching as the set of odd edges of $\mathcal{T}$. Conversely, given a perfect matching M, we shall say that a normal odd partition $\mathcal{T}$ is *conformal* to M whenever M is the set of odd edges of $\mathcal{T}$. Assume that a normal odd partition $\mathcal{T}'$ is obtained from the normal odd partition $\mathcal{T}$ by an operation of switching such that the associated perfect matching M remains unchanged then we shall say that we have proceed to a *conformal (to M) switching*. This operation of conformal switching is not always possible on a vertex. Indeed, assume that v is an internal vertex in $T \in \mathcal{T}$ and an end vertex of this trail then when $T(v, v)$ contains the edge of M incident to v, the conformal switching is not allowed since we would obtain a cycle in the transformation (see Figure 2.1,

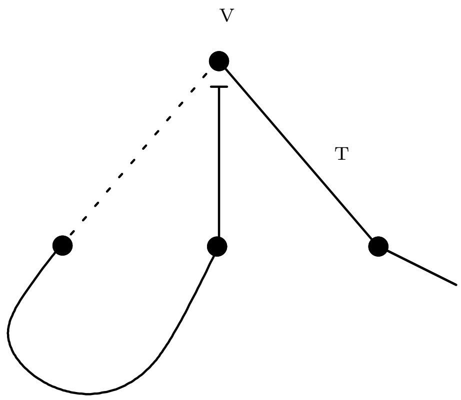


Fig. 1. Conformal switching on v not allowed

the edge of M is the dashed edge). In Theorem 7 we investigate the number of equivalence classes with respect to the conformal switching operation for a given perfect matching M, it turns out that except for the cubic graph on two vertices (see Figure 2.1) the conformal normal partitions are all equivalent.

Fig. 2. Two non equivalent conformal partitions for the cubic graph on 2 vertices

Theorem 7 *Let G be a cubic graph of order at least 4 and M be a perfect matching. Then any two normal partitions $\mathcal{T}$ and $\mathcal{T}'$ conformal to M are conformal switching equivalent.*

Proof

Assume that we have chosen two normal odd partitions $\mathcal{T}$ and $\mathcal{T}'$ which are not conformal switching equivalent and such that $A_{\mathcal{T}\mathcal{T}'}$ has maximum cardinality.

In particular, we have $A_{\mathcal{T}\mathcal{T}'} \neq V(G)$.

Let $v \notin A_{\mathcal{T}\mathcal{T}'}$ and let u_1, u_2 and u_3 be its neighbors. W.l.o.g. we can assume that vu_1 is an edge of M and $e_{\mathcal{T}}(v) = vu_2$ while $e_{\mathcal{T}'}(v) = vu_3$. Since a conformal switch of $\mathcal{T}$ in v leads to a conformal normal partition $\mathcal{T}$" where $e_{\mathcal{T}"}(v) = e_{\mathcal{T}'}(v)$ while nothing is changed elsewhere, we can suppose that this conformal switch is not allowed in v. In the same way a conformal switch of $\mathcal{T}'$ in v is not allowed as well. Hence v is an internal vertex of $T \in \mathcal{T}$ and an end vertex of this trail. Symmetrically, v is an internal vertex of $T' \in \mathcal{T}'$ and an end vertex of this trail (see Figure 2.1). We suppose that y is the second end vertex of T and y' the second end vertex of T'.

Claim 1 $u_2 \neq u_3$

Proof If $u_2 = u_3$ then we have $e_{\mathcal{T}}(u_3) = u_3 v$ (formally we need to still

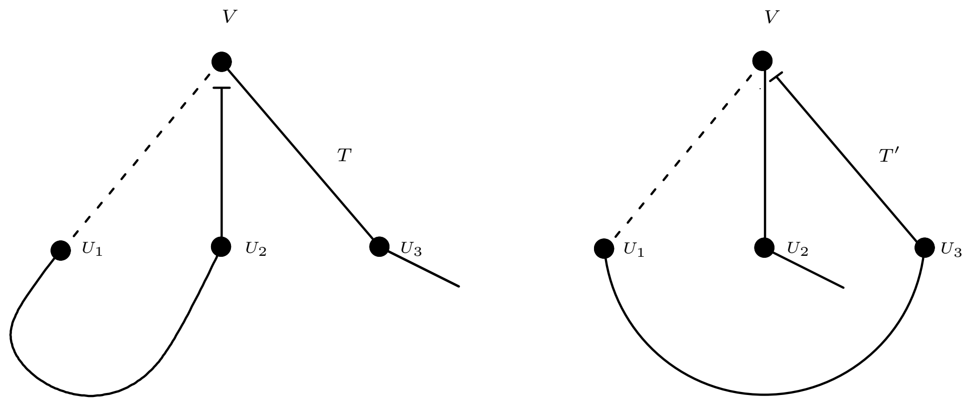


Fig. 3. Conformal switchings on v are not allowed in $\mathcal{T}$ and $\mathcal{T}'$.

distinguish u_2 and u_3) and $e_{\mathcal{T}'}(u_2) = u_2v$. Hence $u_2 \notin A_{\mathcal{T}\mathcal{T}'}$. Let us denote u_2u_4 the edge of M incident to u_2 ($= u_3$) being on the subtrail of $\mathcal{T}$ joining u_1 to u_2 (identical to the the subtrail of $\mathcal{T}'$ joining these two vertices).

We may assume that $u_4 \neq u_1$ otherwise G would be of order 2, a contradiction.

But now, conformal switchings of $\mathcal{T}$ in u_4, u_2 and v lead to a normal partition $\mathcal{T}$" conformal switching equivalent to $\mathcal{T}$. Whatever u_4 belongs or not to $A_{\mathcal{T}\mathcal{T}'}$, $A_{\mathcal{T}"\mathcal{T}'}$ has more vertices than $A_{\mathcal{T}\mathcal{T}'}$, a contradiction. ∎

From now on, we can assume that $u_2 \neq u_3$.

Claim 2 $u_1 \neq u_2$

Proof Assume not, thus $u_1 \notin A_{\mathcal{T}\mathcal{T}'}$. From Claim 1 we have $u_1 \neq u_3$.

We have $y = u_1$, otherwise we can transform $\mathcal{T}$ in $\mathcal{T}''$ by using a conformal switching on u_1 followed by a conformal switching in v and we obtain $|A_{\mathcal{T}''\mathcal{T}'}| > |A_{\mathcal{T}\mathcal{T}'}|$, a contradiction.

But now, conformal switchings of $\mathcal{T}$ in u_3, u_1 and v lead to a normal partition $\mathcal{T}$" conformal switching equivalent to $\mathcal{T}$. Whatever u_3 belongs or not to $A_{\mathcal{T}\mathcal{T}'}$, $A_{\mathcal{T}"\mathcal{T}'}$ has more vertices than $A_{\mathcal{T}\mathcal{T}'}$, a contradiction. ∎

Similarly $u_1 \neq u_3$. If we consider the subtrails $T(u_1, u_2)$ there is a certainly a vertex on that trail for which the associated marked edge $e_{\mathcal{T}}(w) \neq e_{\mathcal{T}'}(w)$. Assume that w is the first vertex with that property when running from u_1 to u_2 on T (let us remark that $T(u_1, w) = T'(u_1, w)$). Hence $w \notin A_{\mathcal{T}\mathcal{T}'}$. Let x be the neighboring vertex of w on $T(v, w)$ (it may happens that $u_1 = w$, in that case $x = v$) and let Q be the trail of $\mathcal{T}$ ending in w with the marked edge $e_{\mathcal{T}}(w)$.

Claim 3 $xw \in M$

Proof If $xw \notin M$ then a conformal switching on w leads to $\mathcal{T}$"

$$\mathcal{T}\text{"} = \mathcal{T} - \{T, Q\} + \{T(y, w), Q + T(w, v)\}$$

Whatever is the status of w relatively to $A_{\mathcal{T}\text{"}\mathcal{T}'}$ we can now perform a conformal switching in v in order to get a new conformal normal partition $\mathcal{T}'''$ for which $A_{\mathcal{T}'''\mathcal{T}'}$ contains at least one vertex more. Indeed let $R \in \mathcal{T}$" be the trail containing v as an internal vertex and $S \in \mathcal{T}$" be the trail containing v as an end vertex then R ends in w and S ends with vu_2 ($e_{\mathcal{T}\text{"}}(v) = vu_2$). Hence the conformal switch on v is now allowed, a contradiction. ∎

Claim 4 $w = y$

Proof Assume that $w \neq y$. Since by Claim 3 $xw \in M$, a conformal switching on w is possible. This leads to the conformal partition $\mathcal{T}$":

$$\mathcal{T}\text{"} = \mathcal{T} - \{T, Q\} + \{T(y, w) + Q, T(w, v)\}$$

In $\mathcal{T}$" v is now on $T(y, w) + Q$ as an internal vertex and $T(w, v)$ as an end vertex. A conformal switch is now allowed on v and we get $\mathcal{T}'''$ such that $A_{\mathcal{T}'''\mathcal{T}'}$ contains one vertex more than $A_{\mathcal{T}\mathcal{T}'}$ (recall that we do not care about w since $w \notin A_{\mathcal{T}\mathcal{T}'}$), a contradiction. ∎

In the same way, we certainly have $w = y'$ (take $\mathcal{T}'$ instead of $\mathcal{T}$). Since by Claim 1 $u_2 \neq u_3$, we must have either $u_2 \neq w$ or $u_3 \neq w$. In considering $\mathcal{T}$, we can decide, w.l.o.g. that $u_2 \neq w$ (if not, we consider $\mathcal{T}'$ where the roles of u_2 and u_3 are exchanged).

In $\mathcal{T}$, u_2 is an internal vertex of T and an end vertex of a trail S with $S \neq T$. A conformal switch is allowed in u_2 and this switch leads to the conformal partition:

$$\mathcal{Q} = \mathcal{T} - \{T, S\} + \{T(y, u_2) + S, u_2v\}$$

And we have $A_{\mathcal{Q}\mathcal{T}'} = A_{\mathcal{T}\mathcal{T}'}$ or $A_{\mathcal{Q}\mathcal{T}'} = A_{\mathcal{T}\mathcal{T}'} - u_2$. But now we can perform a conformal switch on v followed by a conformal switch on w. The first switch on v leads to $\mathcal{R}$

$$\mathcal{R} = \mathcal{Q} - \{T(y, u_2) + S, vu_2\} + \{S + T(u_2, v) + vu_2, T(v, w)\}$$

The second switch on w leads to $\mathcal{S}$

$$\mathcal{S} = \mathcal{R} - \{S + T(u_2, v) + vu_2, T(v, w)\} + \{u_2v + T(v, w) + T(w, v), T(w, u_2) + S\}$$

We have now v and w in $A_{\mathcal{S}\mathcal{T}'}$. Since this set has at least one vertex more than $A_{\mathcal{T}\mathcal{T}'}$, we have a contradiction. □

2.2 Miscellaneous

The following proposition will be essential in the next section.

Proposition 8 *Let G be a cubic graph having 3 normal partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" and $e = xy \in E(G)$ such that x and y are not in A then one of the followings is true:*

- *e is an internal edge in exactly one partition*
- *e is an internal edge in exactly two partitions*

Moreover, in the second case, the edge e itself is a trail of the third partition.

Proof If e is not an internal edge in $\mathcal{T}$, $\mathcal{T}'$ or $\mathcal{T}$" then e is an end edge for a trail of $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$". In x or y we should have two partitions (say $\mathcal{T}$ and $\mathcal{T}'$) for which $e_{\mathcal{T}}(x) = e_{\mathcal{T}'}(x)$ ($e_{\mathcal{T}}(y) = e_{\mathcal{T}'}(y)$ respectively), a contradiction. If e is an internal edge in $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$". Let a and b the two other neighbors of x. We should have then

- $e_{\mathcal{T}}(x) = xa$ or xb
- $e_{\mathcal{T}'}(x) = xa$ or xb
- $e_{\mathcal{T}"}(x) = xa$ or xb

which is impossible. Assume now that e is an internal edge of a trail in $\mathcal{T}$ and in $\mathcal{T}'$ and let a and b the two other neighbors of x. Up to the names of vertices we have

- $e_{\mathcal{T}}(x) = xa$
- $e_{\mathcal{T}'}(x) = xb$

From the third partition T", we must have $e_{\mathcal{T}"}(x) = xy$. In the same way we should obtain $e_{\mathcal{T}"}(y) = yx$. Hence the trail containing $e = xy$ is reduced to e, as claimed. □

Given a normal partition $\mathcal{T}$ $\mu(\mathcal{T})$ is the mean length of the trails in $\mathcal{T}$ and $n_{\mathcal{T}}(i)$ is the number of trails of length i.

Proposition 9 *[4] Let $\mathcal{T}$ be a normal partition of a cubic graph G on n vertices. Then*

- $\mu(\mathcal{T}) = 3$
- $\sum_{i=1}^{i=n+1}(3-i)n_{\mathcal{T}}(i) = 0$

Hence a normal partition whose mean length is 3 has all its trail of length 3.

It can be pointed out that cubic graphs with three compatible normal odd partitions are bridgeless.

Proposition 10 *Let G be a cubic graph with three compatible normal odd partitions then G is bridgeless.*

Proof Assume that xy is a bridge of G and let C be the connected component of $G - xy$ containing x. Since G has three compatible normal odd partitions, one of these partitions, say $\mathcal{T}$, is such that $e_{\mathcal{T}}(x) = xy$. The edges of C are thus partitioned into odd trails (namely the trace of $\mathcal{T}$ on C). We have

$$m = |E(C)| = \frac{3(|C|-1)+2}{2}$$

and m is even whenever $|C| \equiv 3 \ mod \ 4$ while m is odd whenever $|C| \equiv 1 \ mod \ 4$. The trace of $\mathcal{T}$ on C is a set of $\frac{|C|-1}{2}$ trails and this number is odd when $|C| \equiv 3 \ mod \ 4$ and even otherwise. Hence, when $|C| \equiv 3 \ mod \ 4$ we must have an odd number of odd trails partitioning $E(C)$ but, in that case m is even and when $|C| \equiv 1 \ mod \ 4$ we must have an even number of odd trails partitioning $E(C)$ but, in that case m is odd, contradiction. □

In Figure 2.2, we show K_4 provided with three compatible normal odd partitions. Let us remark that, following Theorem 11, we need to have trails of length 5 in at least one partition.

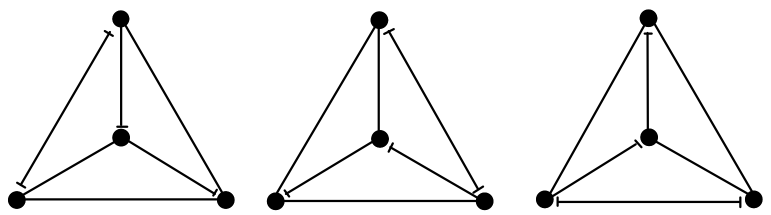


Fig. 4. K_4 with three compatible normal odd partitions

3 On cubic graphs with chromatic index 3

In this section the existence of three compatible normal odd partitions in cubic 3-edge colourable graphs is considered.

Theorem 11 *Let G be a cubic graph then the followings are equivalent*

i) *G has three compatible normal odd partitions of length* 3
ii) *G has three compatible normal odd partitions where each edge is an internal edge in exactly one partition*
iii) *G is bipartite*

Proof Assume first that G can be provided with three compatible normal odd partitions of length 3 $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" . Since the mean length of each partition is 3 (Proposition 9), each trail of each partition has length exactly 3. $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" are thus three normal odd partitions and from Proposition 8, each edge is the internal edge of one trail in exactly one partition. Conversely assume that G can be provided with three compatible normal odd partitions where each edge is an internal edge in exactly one partition, then from Proposition 8 there is no trail of length 1 in any of these partitions. Since the mean length of each partition is 3, that means that each trail in each partition has length exactly 3. Hence $i) \equiv ii)$.

We prove now that $i) \equiv iii)$. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" three compatible normal odd partitions of length 3. Following the proof of Theorem 7 the internal edges of trails of $\mathcal{T}$ ($\mathcal{T}'$ and $\mathcal{T}$" respectively) constitute a perfect matching (say M M' and M" respectively).

Let $a_0a_1a_2a_3$ be a trail of $\mathcal{T}$ and let b_1 and b_2 the third neighbors of a_1 and a_2 respectively. By definition, we have $e_{\mathcal{T}}(a_1) = a_1b_1$ and $e_{\mathcal{T}}(a_2) = a_2b_2$.

Since a_0a_1 and a_2a_3 must be internal edges in a trail of $\mathcal{T}'$ or (exclusively) $\mathcal{T}$", assume w.l.o.g. that a_0a_1 is an internal edge of a trail T'_1 of $\mathcal{T}'$. T'_1 does not use a_1a_2 otherwise $e_{\mathcal{T}'}(a_1) = a_1b_1$, a contradiction with $e_{\mathcal{T}}(a_1) = a_1b_1$ since $\mathcal{T}$ and $\mathcal{T}'$ are compatible. Hence T' uses a_1b_1 and $e_{\mathcal{T}'}(a_1) = a_1a_2$.

Assume now that a_2a_3 is an internal edge of a trail T'_2 of $\mathcal{T}'$. Reasoning in the same way, ge get that $e_{\mathcal{T}'}(a_2) = a_2a_1$. These two results leads to the fact that a_2a_3 must be a trail in $\mathcal{T}'$, which is impossible since each trail has length exactly 3.

Hence, whenever a_0a_1 is supposed to be an internal edge in a trail of $\mathcal{T}'$, we must have a_2a_3 as an internal edge in a trail of $\mathcal{T}$". The two internal vertices of $a_0a_1a_2a_3$ can be thus distinguished, following the fact that the end edge to whom they are incident is internal in $\mathcal{T}'$ (say *red* vertices) or $\mathcal{T}$" (say *blue* vertices). The same holds for each trail in $\mathcal{T}$ (and incidently for each partition $\mathcal{T}'$ and $\mathcal{T}$"). We can remark now that a_1b_2 is an end edge of a trail in $\mathcal{T}$. This end edge cannot be an internal edge in T' since the trail of length 3 going through a_0a_1 ends with a_1b_1. Hence a_1b_1 is an internal edge in $\mathcal{T}$" and b_1 is a blue vertices. Considering now a_0, this vertex is the internal vertex of a trail of length 3 of $\mathcal{T}$. Since $a_0a_1 \in M'$ and M' is a perfect matching, a_0 cannot be incident to an other internal edge of a trail in $\mathcal{T}'$ and a_0 must be a blue vertex. Hence a_1 is a red vertex and its neighbors are all blue vertices. Since we can perform this reasoning in each vertex G is bipartite as claimed.

Conversely, assume that G is bipartite and let $V(G) = \{W, B\}$ be the bipartition of its vertex set. In the following, a vertex in W will be represented by a circle ($\circ$) while a vertex in B will be represented by a bullet ($\bullet$). From König's theorem [7] G is a cubic 3-edge colourable graph . Let us consider a coloring of its edge set with three colors $\{\alpha, \beta, \gamma\}$. Let us denote by $\alpha \bullet \beta \circ \gamma$ a trail of length 3 which is obtained in considering an edge uv ($u \in B$ and $v \in W$) colored with β together with the edge colored α incident with u and the edge colored with γ incident with v. It can be easily checked that the set $\mathcal{T}$ of $\alpha \bullet \beta \circ \gamma$ trails of length 3 is a normal odd partition of length 3. We can define in the same way $\mathcal{T}'$ as the set of $\beta \bullet \gamma \circ \alpha$ trails of length 3 and $\mathcal{T}$" as the set of $\gamma \bullet \alpha \circ \beta$ trails of length 3.

Hence $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" is a set of three normal odd partitions of length 3. We claim that these partitions are compatible. Indeed, let $v \in W$ be a vertex and u_1, u_2 and u_3 its neighbors. Assume that u_1v is colored with α, u_2v is colored with β and u_3v is colored with γ . Hence u_1v is internal in an $\gamma \bullet \alpha \circ \beta$ trail of $\mathcal{T}$" and $e_{\mathcal{T}"}(v) = vu_3$. The edge u_2v is internal in an $\alpha \bullet \beta \circ \gamma$ trail of $\mathcal{T}$ and $e_{\mathcal{T}}(v) = vu_1$. The edge u_3v is internal in an $\beta \bullet \gamma \circ \alpha$ trail of $\mathcal{T}'$ and $e_{\mathcal{T}'}(v) = vu_2$. Since the same reasoning can be performed in each vertex of G, the three $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" partitions are compatible.

$\square$

Theorem 12 *Let G be a cubic graph with three compatible normal odd partitions $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" such that*

- *$\mathcal{T}$ has length 3*
- *$\mathcal{T}'$ and $\mathcal{T}$" are odd*

Then G is a cubic 3-edge colourable graph.

Proof Since $\mathcal{T}$ has length 3, every trail of $\mathcal{T}$ has length 3 (see Proposition 9). Hence there is no edge which can be an internal edge of a trail of $\mathcal{T}'$ and a trail of $\mathcal{T}$", since, by Proposition 8 such an edge would be a trail of length 1 in $\mathcal{T}$. The perfect matchings associated to $\mathcal{T}'$ and $\mathcal{T}$" (see Theorem 7) are thus disjoint and induce an even 2-factor of G, which means that G is a cubic 3-edge colourable graph, as claimed. $\square$

Let G be a simple cubic 3-edge colourable graph with a proper 3-edge colouring using colours in $\{\alpha, \beta, \gamma\}$. According to Theorem 7 above we know that G has 3 odd normal partitions $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$ such that $\mathcal{T}_\alpha$ is conformal to the perfect matching of edges coloured with α, $\mathcal{T}_\beta$ is conformal to β and $\mathcal{T}_\gamma$ is conformal to γ. Let v be a vertex, we shall denote by v_α, v_β and v_γ the **edge**, coloured α, β and γ respectively, incident with v.

Lemma 13 *Let G be a cubic 3-edge colourable graph with a proper 3-edge colouring $\{\alpha, \beta, \gamma\}$. Assume that G has three compatible normal odd partitions $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$ which are, respectively, conformal to α, β and γ. Then G' obtained from G in subdividing an edge e with two vertices u and v (u adjacent to x and v adjacent to y) and joining these two vertices by an additional edge has also this property.*

Proof Assume that $e = xy$ is coloured with α We get a proper 3-edge colouring of G' as follows : let the edges xu and vy be colored α while the two other edges incident to u and v are respectively coloured β and γ. By proposition 9, e is an internal edge in at least one normal partition (say $\mathcal{T}_\alpha$) and an end edge in at least one other (say $\mathcal{T}_\beta$). We are going to transform the normal odd partition $\mathcal{T}_\mu$ of G, conformal to μ, into the normal odd partition $\mathcal{T}'_\mu$ of G', conformal to $\mu \in \{\alpha, \beta, \gamma\}$.

Let $T_\alpha \in \mathcal{T}_\alpha$ be the trail containing e. In order to get the normal odd partition $\mathcal{T}'_\alpha$ we replace the sub-trail xey of T_α with $xx_\alpha uu_\beta vv_\alpha y$ and we add the trail of length 1, $uu_\gamma v$. Obviously all the trails of $\mathcal{T}'_\alpha$ have odd edges of colour α.

e is an end-edge of some trail of $\mathcal{T}_\beta$. W.l.o.g. we can consider that x is its end vertex. Let us denote $T_{\beta,x}$ this trail. We replace the sub-trail xey of $T_{\beta,x}$ with $vv_\gamma uu_\beta vv_\alpha y$ and we consider the trail of length 1, $xx_\alpha u$. We get hence a normal odd partition T'_β of G' conformal with β.

It remains two cases to consider.

case 1: e is an end-edge of some trail of $\mathcal{T}_\gamma$

Assume that y is its end vertex and let us denote this trail $T_{\gamma,y}$. We replace the subtrail xey of $T_{\gamma,y}$ with $xx_\alpha uu_\gamma vv_\beta u$ and we consider the trail of length 1 $yy_\alpha v$.

case 2: e is an internal edge of some trail T_γ of $\mathcal{T}_\gamma$. We replace the subtrail xey of $T_{\gamma,y}$ with $xx_\alpha uu_\gamma vv_\beta u$ and we consider the trail of length 1 $yy_\alpha v$.

In any case, we get a normal odd partition T'_γ of G' conformal with γ and we can check that these 3 normal odd partitions $\mathcal{T}'_\alpha, T'_\beta$ and T'_γ of G' are compatible as expected. $\square$

Lemma 14 *Let G be a cubic 3-edge colourable graph with a proper 3-edge colouring. $\{\alpha,\beta,\gamma\}$. Assume that G has three compatible normal odd partitions $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$ which are, respectively, conformal to α, β and γ. Then G' obtained from G in expanding a vertex by a triangle has also this property.*

Proof Let us expand a vertex x of G by a triangle say abc. We colour the edges ab, ac, and bc in order to get a proper 3-edge colouring in G'. Assume w.l.o.g. that the edge ab (resp. bc, ac) is coloured β (resp. α, γ).

We suppose that $e_{\mathcal{T}_\alpha}(x) = x_\gamma$. The vertex x is an internal vertex of some trail in $\mathcal{T}_\alpha$, say $T_{\alpha,x}$, we replace x in $T_{\alpha,x}$ with $aa_\beta bb_\alpha c$ and we add the trail of length 1, $aa_\gamma c$. Thus we get a normal odd partition of the edge set of G' whose all trails have odd edges of colour α.

We proceed in a similar way for $\mathcal{T}'_\beta$ and $\mathcal{T}_\gamma$ and we get three compatible normal odd partitions with the desired property. $\square$

Let μ be some colour in $\{\alpha,\beta,\gamma\}$, we denote $T_{\mu,v}$ the trail of $\mathcal{T}_\mu$ for which v is an internal vertex while $T'_{\mu,v}$ denotes the trail of $\mathcal{T}_\mu$ ending in v. Since the odd edges of $T_{\mu,v}$ have colour μ we denote the endpoints of this trail as $a_{T_{\mu,v}}$ and $b_{T_{\mu,v}}$ in such a way that the sub-trail of $T_{\mu,v}$ whose endpoints are $a_{T_{\mu,v}}$ and v contains v_μ as an end-edge. Thus we write $T_{\mu,v} = a_{T_{\mu,v}} \dots v_\mu v \dots b_{T_{\mu,v}}$. In addition $b_{T'_{\mu,v}}$ denotes the endpoint of $T'_{\mu,v}$ distinct from v.

Lemma 15 *Let G be a simple cubic 3-edge colourable graph with a proper 3-edge colouring using colours in $\{\alpha,\beta,\gamma\}$. Let $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$ three normal odd partitions conformal to α, β and γ and $A = A_{\mathcal{T}_\alpha\mathcal{T}_\beta} + A_{\mathcal{T}_\beta\mathcal{T}_\gamma} + A_{\mathcal{T}_\gamma\mathcal{T}_\alpha}$ has minimal size. Let μ,ν,ξ be 3 distinct colors of $\{\alpha,\beta,\gamma\}$.*

If $x \in A_{\mathcal{T}_\mu\mathcal{T}_\nu}$, x_μ is an end edge of $T'_{\xi,x}$ and y is the neighbor of x such that $x_\xi = y_\xi$. Then

- $a_{T_{\mu,x}} = x$,
- $y \notin A$,
- $T'_{\nu,x} = xx_\xi y$,
- $T_{\xi,x} = xx_\mu \dots y_\nu y x_\xi x x_\nu \dots y_\mu y$.

Proof Assume that $a_{T_{\mu,x}} \neq x$. We use an odd switch of $\mathcal{T}_\mu$ on x and we get a normal odd partition $\mathcal{T'}_\mu$ as follows :

$$\mathcal{T'}_\mu = \mathcal{T}_\mu - \{T_{\mu,x}, T'_{\mu,x}\} \cup \{a_{T_{\mu,x}} \dots x_\mu x x_\xi \dots b_{T'_{\mu,x}}, xx_\nu \dots b_{T_{\mu,x}}\}$$

Observe that the trails $a_{T_{\mu,x}} \dots x_\mu x x_\xi \dots b_{T'_{\mu,x}}$ and $xx_\nu \dots b_{T_{\mu,x}}$ of $\mathcal{T'}_\mu$ are odd and their odd edges of colour μ. Moreover $e_{\mathcal{T'}_\mu}(x) = x_\nu$, consequently we have

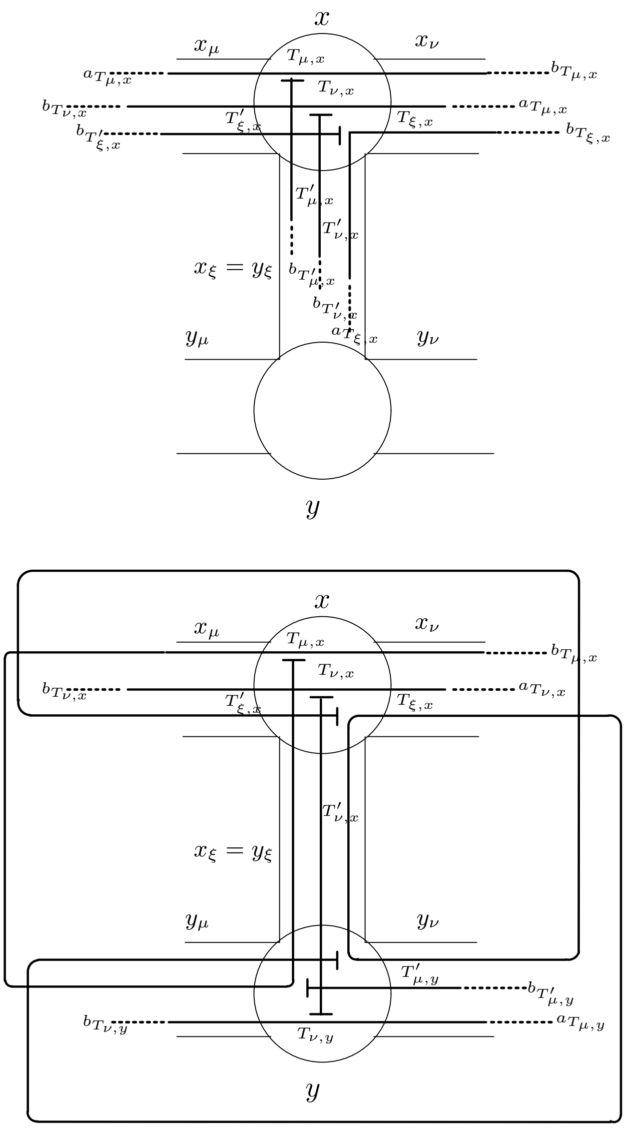


Fig. 5. Situation in Lemma 15

a contradiction with $A_{\mathcal{T}'_\mu \mathcal{T}_\nu} \cup A_{\mathcal{T}_\nu \mathcal{T}_\xi} \cup A_{\mathcal{T}'_\mu \mathcal{T}_\xi}$ whose size is $|A|-1$. Thus $a_{T_{\mu,x}} = x$ and $T_{\mu,x} = T'_{\mu,x}$.

Assume that $y \in A$. Thus whatever the odd switches we will perform on y, this will not increase the size of A. We get a normal odd partition $\mathcal{T}'_\mu$ from $\mathcal{T}_\mu$ in using odd switches of $\mathcal{T}_\mu$ on y and x. More precisely we write

$$\mathcal{T}'_\mu = \mathcal{T}_\mu - \{T_{\mu,x}\} \cup \{yy_\mu \dots x_\mu x x_\xi y y_\nu \dots b_{T'_{\nu,y}}, xx_\nu \dots b_{T_{\mu,x}}\}.$$

Once again, when performing those operations we get 3 odd normal partitions which are compatible on x, a contradiction with the the fact that A has minimal size.

Since $T_{\mu,x}$ contains the edge y_μ, the vertex y and ends on x the edge y_ν must be an end-edge of $T'_{\mu,y}$. moreover y not being in A, the trail $T_{\xi,y}$ must contain the edge y_ν and thus y_μ is an end-edge of $T'_{\xi,y}$, it follows that x_ξ is an end-edge of $T'_{\nu,y}$, that is the trail $T'_{\nu,x}$ has length 1 namely $T'_{\nu,x} = xx_\xi y = T'_{\nu,y}$.

If $a_{T_{\xi,y}} \neq y$ we can perform 3 odd switches (of $\mathcal{T}_\mu$, $\mathcal{T}_\nu$ and $\mathcal{T}_\xi$) on y an the odd switch of $\mathcal{T}_\mu$ on x, in other words we write :

$$\mathcal{T'}_\mu = \mathcal{T}_\mu - \{T_{\mu,x}\} \cup \{yy_\mu \dots x_\mu xxx_\xi yy_\nu \dots b_{T'_{\mu,y}}\}$$
$$\mathcal{T'}_\nu = \mathcal{T}_\nu - \{T_{\nu,y}, xx_\xi y\} \cup \{b_{T_{\nu,y}} \dots y_\mu y, xx_\xi yy_\nu \dots a_{T_{\nu,y}}\}$$
$$\mathcal{T'}_\xi = \mathcal{T}_\xi - \{T_{\xi,y}, T'_{\xi,y}\} \cup \{b_{T'_{\xi,y}} \dots y_\mu yy_\xi xx_\nu \dots a_{T_{\xi,y}}, yy_\nu \dots b_{T_{\xi,y}}\}$$

But now, $A_{\mathcal{T'}_\mu \mathcal{T'}_\nu} \cup A_{\mathcal{T'}_\nu \mathcal{T'}_\xi} \cup A_{\mathcal{T'}_\mu \mathcal{T'}_\xi}$ has less vertices than A, a contradiction. Thus $T_{\xi,y} = T_{\xi,x} = yy_\mu \dots x_\nu xx_\xi yy_\nu \dots b_{T_{\xi,y}}$. When $b_{T_{\xi,y}} \neq x$ we perform the odd switches of $\mathcal{T}_\nu$ and $\mathcal{T}_\xi$ on x and we get 2 new normal odd partitions, that is :

$$\mathcal{T'}_\nu = \mathcal{T}_\nu - \{T_{\nu,x}, xx_\xi y\} \cup \{yx_\xi xx_\nu \dots a_{T_{\nu,x}}, xx_\mu \dots b_{T_{\nu,x}}\}$$

if $a_{T_{\nu,x}} \neq x$ or

$$\mathcal{T'}_\nu = \mathcal{T}_\nu - \{T_{\nu,x}, xx_\xi y\} \cup \{yx_\xi xx_\nu \dots x_\mu x\}$$

otherwise.

In addition we set

$$\mathcal{T'}_\xi = \mathcal{T}_\xi - \{T_{\xi,y}, T'_{\xi,x}\} \cup \{b_{T'_{\xi,x}} \dots x_\mu xx_\xi yy_\nu \dots b_{T_{\xi,y}}\},$$

it follows that $\mathcal{T}_i$,$\mathcal{T'}_\nu$ and $\mathcal{T'}_\xi$ are compatible on x, a contradiction. Consequently $T_{\xi,x} = xx_\mu \dots y_\nu yx_\xi xx_\nu \dots y_\mu y$ which proves the Lemma. □

Theorem 16 *Let G be a cubic 3-edge colourable graph with a proper 3-edge colouring. Then G has three compatible normal odd partitions $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ which are conformal respectively to α, β and γ.*

Proof Assume that G has a 3 edge-colouring using colours in $\{\alpha, \beta, \gamma\}$. From Lemma 13 and Lemma 14, we can suppose that G has no multiple edge and no triangle. Let us consider 3 odd normal partitions $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$ such that the odd edges are coloured α, β and γ respectively. Assume that $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ are such that the size of A is minimum. We suppose that $A \neq \emptyset$ otherwise $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ are compatible and we are done.

Let $u \in A$, w.l.o.g we suppose that $u \in A_{\mathcal{T}_\alpha \mathcal{T}_\beta}$. Henceforth, $e_{\mathcal{T}_\alpha}(u) = u_\gamma = e_{\mathcal{T}_\beta}(u)$. Let v be the neighbor of u incident to u_γ.

Claim 1 $a_{T_{\alpha,u}} = u$, $v \notin A$, $T'_{\beta,u} = uu_\xi v$, *and* $T_{\gamma,u} = uu_\alpha \dots v_\beta vu_\gamma uu_\beta \dots v_\alpha v$.

Proof We just apply Lemma 15 on $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ with $u=x$, $v=y$, $\alpha=\mu$, $\beta=\nu$ and $\gamma=\xi$. ∎

Let v_1 and v_2 be the neighbors of v such that v_1 is incident to v_α and v_2 is incident to v_β.

Claim 2 *$v_2 \notin A$ and $b_{T'_{\alpha,v}} = v_2$.*

Proof Recall that $T_{\gamma,u} = uu_\alpha \dots v_\beta v u_\gamma u u_\beta \dots v_\alpha v$ and $T'_{\beta,u} = uu_\gamma v$. In using odd switches of $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ on u and the odd switch of $\mathcal{T}_\gamma$ on v_2 we obtain :

$$\mathcal{T'}_\beta = \mathcal{T}_\beta - \{T'_{\beta,u}, T_{\beta,u}\} \cup \{uu_\alpha \dots b_{T_{\beta,u}}, vu_\gamma u u_\beta \dots a_{T_{\beta,u}}\}$$

$$\mathcal{T'}_\gamma = \mathcal{T}_\gamma - \{T_{\gamma,u}, T'_{\gamma,v_2}\} \cup \{uu_\beta \dots v_{1\gamma} v_\alpha v, u_2 v_\beta v u_\gamma u u_\alpha \dots v_2 \dots b_{T'_{\gamma,v_2}}\}$$

If $v_2 \in A$, we have $A_{\mathcal{T}_\alpha \mathcal{T}'_\beta} \cup A_{\mathcal{T}'_\beta \mathcal{T}'_\gamma} \cup A_{\mathcal{T}_\alpha \mathcal{T}'_\gamma} = A - \{u\}$, a contradiction with the choice of $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$.

Consequently, the edge v_β being an internal edge of $T_{\beta,v} \in \mathcal{T}_\beta$ and an internal edge of $T_{\gamma,u} \in \mathcal{T}_\gamma$ and since $v, v_2 \notin A$, by Proposition8, the trail $T'_{\alpha,v}$ has length 1, we can write $T'_{\alpha,v} = vv_\beta v_2$, that is $b_{T'_{\alpha,v}} = v_2$. ∎

Let us use odd switches of $\mathcal{T}_\alpha$ on u and v as well as an odd switch of $\mathcal{T}_\beta$ on v, in other words we get the odd normal partitions :

$$\mathcal{T'}_\alpha = \mathcal{T}_\alpha - \{T_{\alpha,u}, T'_{\alpha,v}\} \cup \{uu_\beta \dots b_{T_{\alpha,u}}, vv_\gamma u u_\alpha \dots v_1 v_\alpha v v_\beta \dots b_{T'_{\alpha,v}}\}$$

and

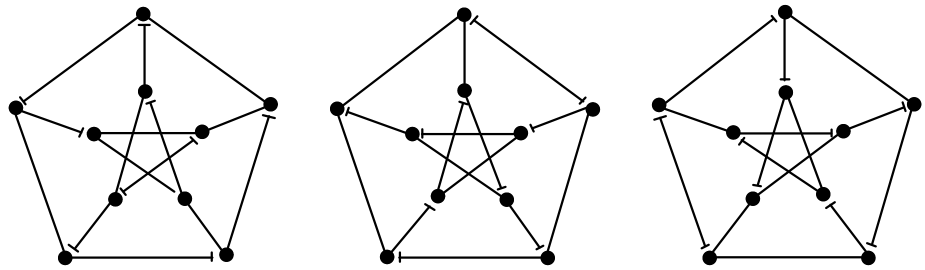
$$\mathcal{T'}_\beta = \mathcal{T}_\beta - \{T_{\beta,v}, T'_{\beta,v}\} \cup \{uu_\gamma v v_\beta \dots a_{T_{\beta,v}}, vv_\alpha \dots b_{T_{\beta,v}}\}$$

But now, we apply Lemma 15 with $x=v$, $y=v_1$, $\mu=\gamma$, $\nu=\beta$,$\xi=\alpha$ and with $A' = A_{\mathcal{T}'_\alpha \mathcal{T}'_\beta} \cup A_{\mathcal{T}'_\beta \mathcal{T}_\gamma} \cup A_{\mathcal{T}'_\alpha \mathcal{T}_\gamma} = A - \{u\} \cup \{v\}$. It follows that the trail of $\mathcal{T'}_\alpha$ having v_α as an internal edge, namely $vv_\gamma u u_\alpha \dots v_1 v_\alpha v v_\beta \dots b_{T'_{\alpha,v}}$ must have v and v_1 as endpoints, a contradiction with Claim 2.

□

4 On cubic graphs with chromatic index 4

A Snark is a bridgeless cubic graph with edge chromatic number of four. By Proposition 10 a cubic graph with three compatible normal odd partitions must be bridgeless. thus in this section we consider the problem of providing three compatible normal odd partitions for some known Snarks as an extension of the Petersen graph as well as the family of Flower Snarks.



(a) Partition $\mathcal{T}_1$ (b) Partition $\mathcal{T}_2$ (c) Partition $\mathcal{T}_3$

Fig. 6. three compatible normal odd partitions of the Petersen graph

4.1 An extension of the Petersen graph

In Figures 6(a), 6(b) and 6(c) we give three compatible normal odd partitions of the Petersen graph. It can be pointed out that these three compatible normal odd partitions are isomorphic. Indeed, we have, in each partition, a path of length 5, 3 paths of lengths 3 and 1 path of length 1. In some sense this shows that Theorem 12 is best possible.

We have seen that, given a cubic graph with three compatible normal odd partitions, we can obtain a new cubic graph with the same property in subdiving an edge with 2 vertices an adding an edge between these two vertices (see Lemma 13) or in expanding a vertex with a triangle (see Lemma 14). We must say that these Lemmas are proved in a more restrictive context since we deal with conformal odd partitions, but it is an easy task to see that these lemmas remain true when we drop this constraint. We do not know whether the converse operation of the expansion (shrinking a triangle into a single vertex) allow us to preserve three compatible normal odd partitions in the resulting graph.

Our purpose now is to construct an example of a cubic graph with chromatic index 4 and three compatible normal odd partitions starting from two copies of the Petersen graph and using a restricted form of the classical *composition* (see Figure 7) operation. It's a folklore's result that such a composition has chromatic index 4 as soon as one of the two graphs involved has itself chromatic index 4. The interesting point is that we have provided the Petersen graph with three compatible normal odd partitions but two of them are odd while the third one contains 2 even trails.

Assume that G and G' are two copies of a cubic graph. In order to be more concise, when O denote an object (a vertex, a trail ...) of G, O' will denote the corresponding object (using the obvious isomorphism) of G'. Let $u \in V(G)$ and assume that $N(u) = \{a, b, c\}$ (we do not claim that these vertices are distinct since we do not consider only simple graphs). The *composition on* $\{u, u'\}$ of G and G' is obtained in deleting u in G and u' in G' and joining

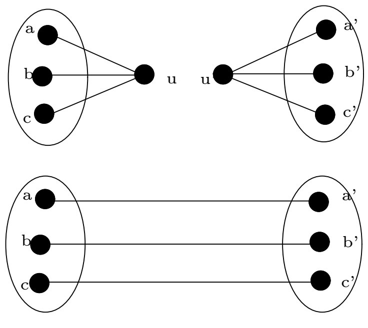
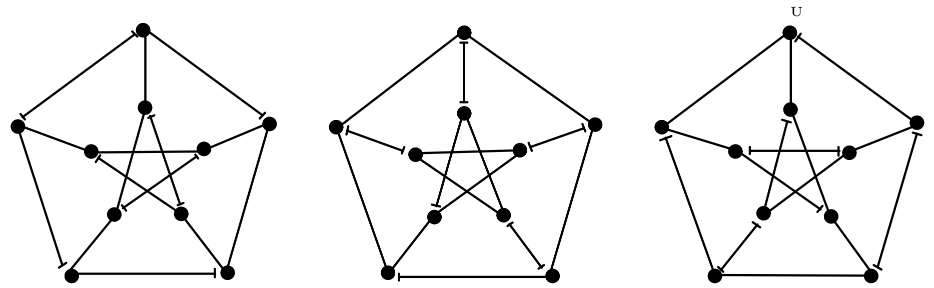


Fig. 7. Composition of two cubic graphs



(a) The odd partition $\mathcal{T}_1$ (b) The odd partition $\mathcal{T}_2$ (c) The partition $\mathcal{T}_3$

Fig. 8. The partitions of the Petersen graph considered in Proposition 18

a to a', b to b' and c to c'. The restriction of the composition invoked above comes from the fact that we impose to join similar vertices.

Proposition 17 *Let G and G' two copies of a cubic graph with $u \in V(G)$ and $u' \in V(G')$. Let H be the cubic graph obtained by composition on $\{u, u'\}$. Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be three compatible normal odd partitions of G and $\mathcal{T}_1'$, $\mathcal{T}_2'$, $\mathcal{T}_3'$ the corresponding normal partitions of G'.*

Assume that $u \notin E_{\mathcal{T}_i}(u)$, for all $i \in \{1, 2, 3\}$ then, there exists three compatible normal odd partitions $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ of H such that

for $i \in \{1, 2, 3\}$

- $\forall v \in V(G-u)$ $e_{\mathcal{Q}_i}(v) = e_{\mathcal{T}_i}(v)$
- $\forall v' \in V(G'-u')$ $e_{\mathcal{Q}_i}(v') = e_{\mathcal{T}'_i}(v')$

Proof Let $i \in \{1, 2, 3\}$, since $u \notin E_{\mathcal{T}_i}(u)$, we have, obviously $u' \notin E_{\mathcal{T}'_i}(u')$. Hence, in $G-u$ the trace of the trails T_1 and T_2 of $\mathcal{T}_i$ using u give three distinct trails A, B and C. It can be noticed that some of them may have length 0. When A, by example, has length 0 this means that the marked edge $e_{T_1}(a)$ associated to a is au. The concatenation of A, aa' and A' is a trail Q_1 of H and we define in the same way Q_2 as the concatenation of B, bb' and B' and Q_3 with C, cc' and C'. Let $\mathcal{Q}_i$ be the partition of H

$$\mathcal{Q}_i = \{\mathcal{T}_i - \{T_1, T_2\}\} \cup \{\mathcal{T}'_i - \{T_1', T_2'\}\} \cup \{Q_1, Q_2, Q_3\}$$

It is a routine matter to check that $\mathcal{Q}_i$ is a normal partition such that $\forall v \in V(G-u)$ $e_{\mathcal{Q}_i}(v) = e_{\mathcal{T}_i}(v)$ since we only extend A, B and C with the edges aa' bb' and cc' in order to get Q_1, Q_2 and Q_3. This operation does not affect the marked edges in $G-u$ (be careful, in the example cited above, when A has length 0, Q_1 is reduced to aa' and the marked edge associated to a is aa' as well as the marked edge associated to a' is $a'a$). The same holds with $e_{\mathcal{Q}_i}(v')$ when $v' \in V(G'-u')$.

Since Q_1 is the concatenation of A, aa' and A' Q_1 has obviously odd length since A and A' are isomorphic trails. Hence Q_1, Q_2 and Q_3 have all odd length.

Thus the partitions $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are three compatible normal odd partitions of H as claimed. $\square$

Proposition 18 *Let H be the graph obtained by composition of two Petersen graphs on $\{u, u'\}$ (where u is any vertex of the Petersen graph). Then H can be provided with three compatible normal odd partitions.*

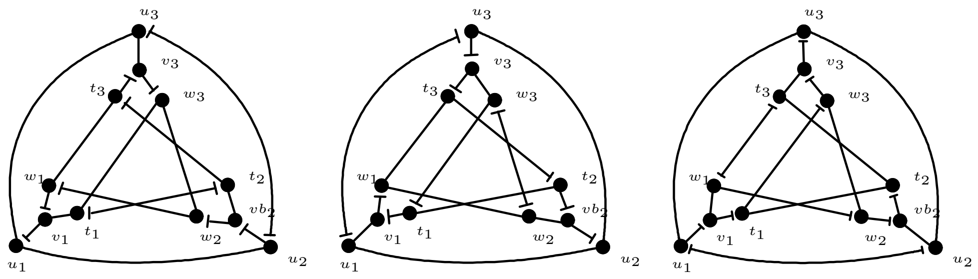
Proof Figures 8(a), 8(b) and 8(c) show that the Petersen graph can be provided with three compatible normal odd partitions $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. In Figures 8(a) and 8(b) we have two odd normal partitions $\mathcal{T}_1$ and $\mathcal{T}_2$ while Figure 8(b) shows a normal partition $\mathcal{T}_3$ with exactly two even trails (of length 8 and 4). These two even trails use u, the one as an internal vertex, the other as an end vertex, moreover, we can check that $u \notin E_{\mathcal{T}_3}(u)$

Using Proposition 17, we see that $\mathcal{T}_1$ and $\mathcal{T}_1'$ give rise to a normal odd partition $\mathcal{Q}_1$ of H, $\mathcal{T}_2$ and $\mathcal{T}_2'$ give a normal odd partition $\mathcal{Q}_2$ while $\mathcal{T}_3$ and $\mathcal{T}_3'$ give a normal partition $\mathcal{Q}_3$. Moreover $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are pairwise compatible.

We claim that $\mathcal{Q}_3$ is odd. Indeed the two even trails of $\mathcal{T}_3$ are now transformed into the trails using aa', bb' and cc' (where $N(u) = \{a, bc\}$) in the Petersen graph). The last item of Proposition 17 insures us that these trails have odd length. By construction of $\mathcal{Q}_3$ the other trails are trails of $\mathcal{T}_3$ or $\mathcal{T}_3'$ which have odd lengths. $\square$

4.2 Flower Snarks

For an odd $k \geq 3$ the Flower Snark F_k (see [6]) is the cubic graph on $4k$ vertices $u_1, u_2, \ldots u_k$, $v_1, v_2, \ldots v_k$, $w_1, w_2, \ldots w_k$, $t_1, t_2, \ldots t_k$ such that $u_1 u_2 \ldots u_k$ is an induced cycle of length k, $w_1 w_2 \ldots w_k t_1 t_2 \ldots t_k$ is an induced cycle of length $2k$ and for $i = 1 \ldots k$ the vertex v_i is adjacent to u_i, w_i and t_i.



(a) partition $\mathcal{T}_1$ (b) partition $\mathcal{T}_2$ (c) partition $\mathcal{T}_3$

Fig. 9. Three compatible normal odd partitions of the Flower Snark F_3

Proposition 19 *Let F_k ($k \geq 3$, k odd) be a Flower Snark, F_k can be provided with three compatible normal odd partitions.*

Proof In Figure 9 we propose three compatible normal odd partitions, namely $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, of the Flower Snark F_k when $k = 3$. Moreover, when considering the vertices u_1, v_1, w_1, t_1, u_2, v_2, w_2, t_2 we have the following situation (recall that $k = 3$):

$$e_{\mathcal{T}_1}(u_1) = u_1v_1,\ e_{\mathcal{T}_1}(v_1) = v_1w_1,\ e_{\mathcal{T}_1}(w_1) = w_1w_2,\ e_{\mathcal{T}_1}(t_1) = t_1t_2 \tag{1}$$
$$e_{\mathcal{T}_2}(u_1) = u_1u_k,\ e_{\mathcal{T}_2}(v_1) = v_1t_1,\ e_{\mathcal{T}_2}(w_1) = w_1v_1,\ e_{\mathcal{T}_2}(t_1) = t_1w_k \tag{2}$$
$$e_{\mathcal{T}_3}(u_1) = u_1u_2,\ e_{\mathcal{T}_3}(v_1) = v_1u_1,\ e_{\mathcal{T}_3}(w_1) = w_1t_k,\ e_{\mathcal{T}_3}(t_1) = t_1v_1 \tag{3}$$

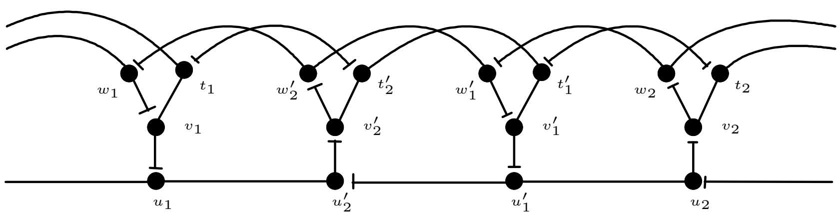
$$e_{\mathcal{T}_1}(u_2) = u_2u_3,\ e_{\mathcal{T}_1}(v_2) = v_2u_2,\ e_{\mathcal{T}_1}(w_2) = w_2v_2,\ e_{\mathcal{T}_1}(t_2) = t_2t_1 \tag{4}$$
$$e_{\mathcal{T}_2}(u_2) = u_2v_2,\ e_{\mathcal{T}_2}(v_2) = v_2t_2,\ e_{\mathcal{T}_2}(w_2) = w_2w_3,\ e_{\mathcal{T}_2}(t_2) = t_2t_3 \tag{5}$$
$$e_{\mathcal{T}_3}(u_2) = u_2u_1,\ e_{\mathcal{T}_3}(v_2) = v_2w_2,\ e_{\mathcal{T}_3}(w_2) = w_2w_1,\ e_{\mathcal{T}_3}(t_2) = t_2v_2 \tag{6}$$

Observe that, among the edges u_1u_2, w_1w_2 and t_1t_2 we have :

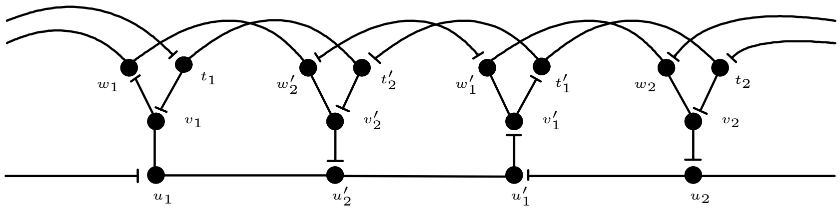
u_1u_2 is the unique odd edge in $\mathcal{T}_1$,
t_1t_2 is the unique odd edge in $\mathcal{T}_2$ and in $\mathcal{T}_3$

Assume by induction that for an odd integer k, $k \geq 3$, the Flower Snark F_k is provided with three compatible normal odd partitions, namely $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. Moreover, assume that the above Properties (1) to (6) are verified by $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

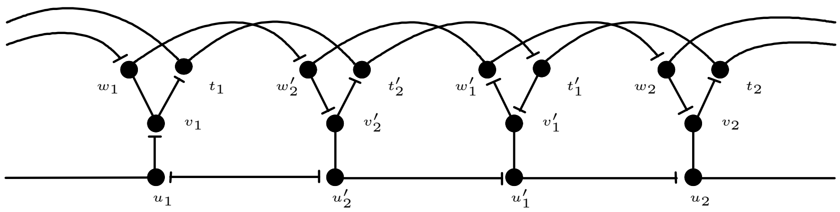
We derive F_{k+2} from F_k as follows :



(a) partition $\mathcal{T}_1'$



(b) partition $\mathcal{T}_2'$



(c) partition $\mathcal{T}_3'$

Fig. 10. Extension of three compatible normal odd partitions from the Flower Snark F_k to the Flower Snark F_{k+2}

$$
\begin{aligned}
V(F_{k+2}) &= V(F_k) \cup \{u_1', v_1', w_1', t_1', u_2', v_2', w_2', t_2'\} \\
E(F_{k+2}) &= E(F_k) - \{u_1u_2, w_1w_2, t_1t_2\} \\
&\qquad \cup \{u_1u_2', w_1w_2', t_1t_2'\} \\
&\qquad \cup \{v_2'u_2', v_2'w_2', v_2't_2'\} \\
&\qquad \cup \{u_2'u_1', w_2'w_1', t_2't_1'\} \\
&\qquad \cup \{v_1'u_1', v_1'w_1', v_1't_1'\} \\
&\qquad \cup \{u_1'u_2, w_1'w_2, t_1't_2\}
\end{aligned}
$$

In other words we add 8 new vertices to F_k, we delete the edges u_1u_2, w_1w_2, t_1t_2 and add new edges in order to obtain the Flower Snark F_{k+2}.

Figure 10(a) 10(b) and 10(c) show three normal partitions of F_{k+2}, $\mathcal{T}_1'$, $\mathcal{T}_2'$ and $\mathcal{T}_3'$, respectively obtained from $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

But now we rename some vertices of F_{k+2} as follows : For $i \geq 2$, u_i, v_i, w_i and t_i are respectively renamed u_{i+2}, v_{i+2}, w_{i+2}, t_{i+2}. The vertices u_1', v_1', w_1' and t_1' are respectively renamed u_3, v_3, w_3 and t_3. The vertices u_2', v_2', w_2' and t_2' are respectively renamed u_2, v_2, w_2 and t_2.

It's a routine matter to check that those partitions are odd, compatible and verify Properties (1) to (6). $\square$

5 Conjectures

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matchings with empty intersection.

Theorem 20 *Let G be a cubic graph with three compatible normal odd partitions then there exist 3 perfect matching M, M' and M" such that $M \cap M' \cap M" = \emptyset$.*

Proof Let M, M' and M" be the associated perfect matchings of $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" respectively. Let v be any vertex and u_1, u_2 and u_3 its neighbors. $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}$" being compatible, we can suppose that $e_{\mathcal{T}}(v) = vu_1$, $e_{\mathcal{T}'}(v) = vu_2$ and $e_{\mathcal{T}"}(v) = vu_3$. vu_1 is an end edge of a trail in $\mathcal{T}$, this edge is not an odd edge in $\mathcal{T}$ and thus $vu_1 \notin M$. In the same way $vu_2 \notin M'$ and $vu_3 \notin M$". Hence, any edge incident to v is contained in at most two perfect matchings among M, M' and M". Which means that $M \cap M' \cap M" = \emptyset$

□

Theorem 20 above implies that the Fan Raspaud Conjecture is true for graphs with three compatible normal odd partitions and we propose as a new conjecture:

Conjecture 21 *Any bridgeless cubic graph can be provided with three compatible normal odd partitions.*

We do not know whether this new conjecture is equivalent to the Fan Raspaud conjecture or not. By the way, the Fan Raspaud conjecture seems to be originated independently by Jackson. Goddyn [5] indeed mentioned this problem proposed by Jackson for $r-$graphs ($r-$regular graphs with an even number of vertices such that all odd cuts have size at least r, as defined by Seymour [8]) in the proceedings of a joint summer research conference on graphs minors which dates back 1991.

Conjecture 22 *[5] There exists $k \geq 2$ such that any r-graph contains $k + 1$ perfect matchings with empty intersection.*

Seymour [8] conjectured that:

Conjecture 23 *[8] If $r \geq 4$ then any r-graph has a perfect matching whose deletion yields an (r-1)-graph.*

Hence Seymour's conjecture leads to a specialized form of Jackson's conjecture when dealing with cubic bridgeless graphs and the Fan Raspaud conjecture appears as a refinement of Jackson's conjecture.

Let $\mathcal{S} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots \mathcal{T}_k\}$ ($k \geq 3$) be a set of odd normal partitions of a cubic graph G. The set $\mathcal{S}$ will be said a *complete system of odd normal partitions of order* k whenever for any vertex v of G there is three partitions in $\mathcal{S}$ which are compatible on v, that is there is $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ in $\mathcal{S}$ such that $e_{\mathcal{T}}(v)$, $e_{\mathcal{T}'}(v)$ and $e_{\mathcal{T}''}(v)$ are three distinct edges.

Conjecture 24 *There exists $k \geq 3$ such that any bridgeless cubic graph has a complete system of odd normal partitions of order at most k.*

If a cubic graph has a complete system of normal odd partitions of order k then it has k perfect matchings with empty intersection. Thus Conjecture 24 implies Conjecture 22 for bridgeless cubic graphs.

References

[1] J. A. Bondy. Basic graph theory: Paths and circuits. In M. Grötschel R.L. Graham and L. Lovász, editors, *Handbook of Combinatorics*, volume 1, pages 3–112. Elsevier, North-Holland, 1995.

[2] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*. Elsevier, North-Holland, 1976.

[3] G. Fan and A. Raspaud. Fulkerson's conjecture and circuit covers. *J. Comb. Theory Ser. B*, 61:133–138, 1994.

[4] J.L. Fouquet and J.M. Vanherpe. On normal partitions in cubic graphs. *preprint*, 2007.

[5] L. Goddyn. Cones, lattices and Hilbert base of circuits and perfect matching. In N. Robertson and P. Seymour, editors, *Graph Structure Theory*, Contemporary Mathematics Volume 147, pages 419–439. American Mathematical Society, 1993.

[6] R. Isaacs. Infinite families of non-trivial trivalent graphs which are not Tait colorable. *Am. Math. Monthly*, 82:221–239, 1975.

[7] D. König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math.Ann*, 77:453–465, 1916.

[8] P. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.(3)*, 38:423–460, 1979.