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Rapport de Recherche

On linear arboricity of cubic graphs

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On linear arboricity of cubic graphs

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Abstract

A *linear forest* is a graph in which each connected component is a chordless path. A *linear partition* of a graph G is a partition of its edge set into linear forests and $la(G)$ is the minimum number of linear forests in a linear partition. When each path has length at most k a linear forest is a *linear k-forest* and $la_k(G)$ will denote the minimum number of linear k-forests partitioning $E(G)$. We clearly have $la_{n-1}(G) = la(G)$.

In this paper we consider linear partitions of cubic simple graphs for which it is well known that $la(G) = 2$. We give a survey of already known results with new ones and new conjectures.

1 Introduction.

For any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. The number of vertices of G, |V(G)|, is denoted by n and the number of edges of G, $|E(G)|$, is denoted by m. If $F \subseteq E(G)$ then $V(F)$ is the set of vertices which are incident to some edges of F. For any path P we shall denote by $l(P)$ the length of P, that is to say the number of its edges. A vertex of degree one of a path P is said to be an *end vertex*, and a vertex of degree two is said to be an *internal* vertex. If u and v are vertices of a path P then $P[u, v]$ denotes the subpath of P end vertices of which are u and v. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced by the vertex-set $V(C)$.
A *linear k-forest* is a graph in which each component is a chordless path of length at most k. The *linear k-arboricity* of an undirected graph G is defined in [18] as the minimum number of linear k-forests needed to partition the set $E(G)$. The linear k-arboricity is a natural refinement of the linear arboricity introduced by Harary [20] (corresponding to linear-$(n-1)$-arboricity). The linear k-arboricity will be denoted by $la_k(G)$.

Let $\chi^{'}(G)$ be the classical chromatic index and let $la(G)$ be the linear arboricity of G. We clearly have:

$$la(G) = la_{n-1}(G) \leq la_{n-2}(G) \leq ... \leq la_2(G) \leq la_1(G) = \chi^{'}(G)$$

We know by Vizing's Theorem [29] that $la_1(G) \leq \Delta(G)+1$ (where $\Delta(G)$ is the maximum degree of G). In [5] it is shown that for any $k \geq 2$, $la_k(G) \leq \Delta(G)$. In an obvious way, for any $k \geq 2$ we have $la_k(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$.
If $\{L_1, \cdots, L_p\}$ is a partition of $E(G)$ in p linear k-forests, and if for any $j \in \{1, \cdots, p\}$, $\omega(L_j)$ denotes the number of maximal paths (or components) of L_j, we have $\mid V(L_j) \mid \leq (k+1)\omega(L_j)$, $\mid E(L_j) \mid = \mid V(L_j) \mid - \omega(L_j)$. Then for any $j \in \{1, \cdots, p\}$ $\mid E(L_j) \mid \leq \frac{k|V(L_j)|}{k+1} \leq \frac{kn}{k+1}$.

Thus $m = \mid E(G) \mid = \mid E(L_1) \mid + \cdots + \mid E(L_p) \mid \leq p\frac{kn}{k+1}$. By choosing $p = la_k(G)$ we obtain $la_k(G) \geq max(\lceil \frac{\Delta(G)}{2} \rceil, \lceil \frac{m(k+1)}{kn} \rceil)$ (see also Proposition 3 in [19]).
In this paper we consider *cubic graphs*, that is to say finite simple 3-regular graphs. Since in a cubic graph $3n = 2m$, the previous formula becomes (Prop. 2 in [5]):

$$la_k(G) \geq max(2, \lceil \frac{3(k+1)}{2k} \rceil)$$

It was shown by Akiyama, Exoo and Harary [1] that $la_{n-1}(G) = 2$ when G is cubic. A natural question ([5]) is the following : what is the smallest integer i, with $2 \leq i \leq n-1$, such that $la_i(G) = 2$? If K_4 denotes the complete graph on 4 vertices then clearly $la_1(K_4) = la_2(K_4) = 3$ and $la_3(K_4) = 2$. We know that there exist two distinct cubic graphs on 6 vertices: PR_3 (two disjoint triangles connected by a matching) and $K_{3,3}$ (the complete bipartite balanced graph on 6 vertices), and it is easy to see that $la_3(PR_3) = 3$, $la_3(K_{3,3}) = 3$, $la_5(PR_3) = 2$ and $la_5(K_{3,3}) = 2$. In [5] Bermond et al. conjectured that $la_5(G) = 2$. Jackson and Wormald [24] proved a weaker version with $k = 18$ instead of 5. Aldred and Wormald [3] proved that 18 can be replaced by 9. Finally, Thomassen [28] proved the conjecture, which is best possible since, in view of $la_4(K_{3,3}) = 3$ and $la_4(PR_3) = 3$, 5 cannot be replaced by 4.

A partition of $E(G)$ into two linear forests L_B and L_R will be called a *linear partition* and we shall denote this linear partition $L = (L_B, L_R)$. We shall say that every path of L_B and every path of L_R is an *unicoloured* path (for instance, a *Blue* path or a *Red* path). For $c \in \{B, R\}$, we shall denote by $E(L_c)$ the set of edges of L_c and by $l(L_c)$ the length of a longest path in L_c. For $c \in \{B, R\}$ let $\omega(L_c)$ be the number of maximal paths (or components) of L_c and we remark that $\omega(L_c) = |V(G)| - |E(L_c)|$. Since every vertex of G is either end vertex of a maximal path of L_B or end vertex of a maximal path of L_R, we have

$$\omega(L_B) \ + \ \omega(L_R) \ = \ \frac{|V(G)|}{2}.$$

An *odd linear forest* is a linear forest each of whose components are paths of odd length. An *odd linear partition* is a partition of $E(G)$ into two odd linear forests. A *semi-odd linear partition* is a linear partition $L = L_B \cup L_R$ such that L_B or L_R is an odd linear forest.

We shall give some complementary results on linear arboricity of cubic graphs and some new results concerning various problems.

2 Jaeger's graphs

A special class of cubic graphs will be considered (Jaeger's graphs in the sequel) and we shall see that these graphs have nice properties leading to new questions (to be developed into forthcoming sections) for the whole set of cubic simple graphs.

2.1 Matchings and transversals of the odd cycles

In [10] Erdős showed how to obtain a large spanning bipartite subgraph of a given simple graph.

Theorem 2.1 *Let G be a simple graph then there is a spanning bipartite subgraph H such that, for every vertex v the degree $d_H(v)$ of v in H verifies $d_H(v) \geq \lceil \frac{d_G(v)}{2} \rceil$.*

By Theorem 2.1, in every cubic graph there is a bipartite spanning subgraph of minimum degree at least 2. This yields the following corollary.

Corollary 2.2 *Let G be a cubic graph, then there is a matching M which is a transversal of the odd cycles.*

Assume that G is a cubic graph and let M be a matching transversal of the odd cycles. Since $G \setminus M$ is bipartite, we can colour $V(G)$ in two colours *Blue* and *Red* accordingly to the bipartition of $G \setminus M$. Let M_B (respectively M_R) be the set of edges of M such that their two end vertices are *Blue* vertices (respectively *Red* vertices). An edge of $E(G)$ is said to be *mixed* when one end is *Blue* while the other is *Red*. Hence, the edges of $G \setminus M$ are mixed while M is partitioned into three sets (some of them, possibly empty)

$$M = M_B + M_R + M'$$

where M' is the subset of mixed edges of M. Note that M_B and M_R induce strong matchings in G.

Theorem 2.3 *A cubic graph is 3-edge colourable if and only if there is a partition of its vertex set into two sets, Blue and Red and a perfect matching M such that every edge in $G - M$ is mixed.*

Proof Let G be a cubic 3-edge colourable graph. Any colour of a 3-edge colouring of G induces a perfect matching M, and the two others colours induce a graph in which each component is an even cycle. Let us colour the vertices of these cycles in *Blue* and *Red* alternately. Hence every edge lying on these cycles is mixed.

Conversely, assume that G has a perfect matching M and a partition of its vertex set into *Blue* and *Red* such that every edge in $G - M$ is mixed. Let us consider the 2-factor of G obtained by deleting M. Since every edge outside M is mixed, this 2-factor is even, which means that G is 3-edge colourable. $\square$

Remark 2.4 Under conditions of theorem 2.3 we certainly have the same number of *Blue* vertices and *Red* vertices, since every edge of the 2-factor $G - M$ is mixed. When considering $M = M_B + M_R + M'$ we have $|M_B| = |M_R|$ since every mixed edge of M uses a vertex in each colour.

2.2 Definitions

Definition 2.5 Let G be a cubic 3-edge colourable graph with a perfect matching M given as in Theorem 2.3 by a 3-edge colouring. We shall say that a partition of M in $M_B + M_R + M'$ is an *associated partition*.

Definition 2.6 A cubic graph G is a *Jaeger's graph* if G contains a perfect matching union of two disjoint strong matchings. A perfect matching which is union of two disjoint strong matchings is said to be a *Jaeger's matching*.

Assume that G is a Jaeger's graph and let M_B and M_R be the two strong matchings which partitions a perfect matching M of G. Let us colour with *Blue* the vertices which are end vertices of edges in M_B and *Red* those which are end vertices of edges in M_R. Since M_B and M_R are strong matchings the remaining edges are mixed. Hence G is 3-edge colourable and, as pointed out in remark 2.4, we have $|M_B| = |M_R| = |M|/2$. The associated partition $M = M_B + M_R + M'$ is such that M' is empty.
In his thesis [23] Jaeger called these cubic graphs *equitable* and pointed out that the above 2-colouring of their vertices leads to a *balanced colouring* as defined by Bondy [7].

2.3 Towards a linear partition

Assume that we are given a cubic 3-edge colourable graph together with an associated partition $M = M_B + M_R + M'$. Let us fix an arbitrary orientation to the cycles of $G \setminus M$. To each vertex v of $V(G)$ we can associate an edge $o(v)$ of $E(G) \setminus M$ such that v is the origin of $o(v)$ with respect to the chosen orientation of the cycle through v. It will be convenient to denote by $s(v)$ (*successor* of v) the end of $o(v)$ in that orientation and by $p(v)$ its *predecessor*. We can colour $o(v)$ in *Blue* or *Red* accordingly to the colour of v. M_B being coloured with *Blue* and M_R with *Red*, we get hence a larger set CL_B of edges coloured with *Blue* (and a set CL_R of edges coloured with *Red*). CL_B and CL_R are linear-forests where each maximal unicoloured path has length 1 or 3. Moreover each edge of $M_B \cup M_R$ is the central edge of a path of length 3. At this point, the only edges which are not coloured are the edges of M' and we do not know how we can affect these edges in CL_B or CL_R in order to get a linear partition of $E(G)$. We shall see in Theorems 2.8 and 2.9 that this can be done for particular cases.

Definition 2.7 We shall refer to the above construction of CL_B and CL_R when an associated partition is given as the *associated linear construction*.

Theorem 2.8 [5] *A cubic graph G has a linear partition $L = (L_B, L_R)$ such that each path has length at most 3 if and only if G is a Jaeger's graph .*

Proof : Suppose that G has a linear partition $L = (L_B, L_R)$ with maximum lengths $l(L_B) \leq 3$ and $l(L_R) \leq 3$. Since $\omega(L_B) + \omega(L_R) = \frac{|V(G)|}{2}$, and $|E(G)| = 3\frac{|V(G)|}{2}$ each path in L_B and L_R have length exactly 3. Let C_B (respectively C_R) be the set of the middle edges of the paths of L_B (respectively L_R). It is an easy task to check that C_B and C_R are strong matchings and $|C_B| = |C_R|$. Moreover $M = C_B \cup C_R$ is a perfect matching and G is a Jaeger's graph .

Conversely, let us suppose that G is a Jaeger's graph and let $M = M_B + M_R$ be an associated partition. Since M' is empty, in using the associated linear construction above, we have coloured every edge of G and each unicoloured path has length 3. □

Theorem 2.9 *Let G be cubic 3-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that M' can be partitioned into two strong matchings M'_B and M'_R. Then there is an odd linear partition of $E(G)$ every maximal path of which has length $1, 3, 5$ or 7.*

Proof : Let CL_B and CL_R be the linear-forests of the associated linear construction. Recall that each maximal path of CL_B (respectively CL_R) has length 1 or 3 and is unicoloured with *Blue* (respectively unicoloured with *Red*). Let $L_B = CL_B \cup M'_B$ and $L_R = CL_R \cup M'_R$, in addition $M = M_B + M_R + M'$ and B denotes the set of *Blue* vertices of G and R its set of *Red* vertices.

We now prove that the components of L_B and L_R are odd paths of length at most 7. We only have to consider components that contain an edge of M'. Without loss of generality, let C be a component of L_B which contains an edge of M'_B.

Claim *Let br be an edge of M'_B such that $b \in B \cap C$ and $r \in R \cap C$ and let $r' = s(b)$. Then the unique neighbour of r' in C is b.*

Proof of Claim Since $G \backslash M$ contains only mixed edges r' is a *Red* vertex. Observe that $o(r')$ is a *Red* edge while the edge of M incident to r' in G, say e, cannot belong to M'_B since M'_B is a strong matching. Moreover, e having a *Red* end must belong either to M_R or to M'_R, consequently e belongs to L_R. Thus, among the three edges incident to r', only $o(b)$ belongs to C and the result follows. ■

Let b_1r_1 be an edge of $C \cap M'_B$ ($b_1 \in B,\ r_1 \in R$). Let us set $r_2 = s(b_1)$. We know by the above Claim that r_2 is a pendant vertex in C.

Let $b_2 = p(r_1)$ and $r_3 = p(b_2)$, obviously $b_2 \in B$, $r_3 \in R$, b_2r_1 is a *Blue* edge and r_3b_2 is a *Red* one. Consider in G the edge of M incident to b_2, say e. M'_B being a strong matching, the edge e cannot belong to M'_B. Moreover, the edge e has a *Blue* end, namely b_2, and thus cannot belong to M_R. If e belongs to

M'_R we are done since e is a *Red* edge and the component C is reduced to the path of length 3 $r_2b_1r_1b_2$.
Assume in the following that e belongs to M_B. From now on e will be denoted b_2b_3 ($b_3 \in B$) and $s(b_3)$ will be denoted r_4, we have $r_4 \in R$. Let e' be the edge of M incident to r_4 in. Since r_4 is a *Red* end of e', e' cannot belong to M_B. If e' is a member of $M_R \cup M'_R$ we are done since e' and $o(r_4)$ both belong to L_R and C is reduced to a path of length 5, namely $r_2b_1r_1b_2b_3r_4$.
Suppose now that $e' \in M'_B$. Let us denote e' as r_4b_4 ($b_4 \in B$) and $s(b_4)$ as r_5. But now, by the above Claim b_4 is the unique neighbour of r_5 in C and thus $C = \{r_2, b_1, r_1, b_2, b_3, r_4, b_4, r_5\}$ induces a path of length 7. □

2.4 Some classes of Jaeger's graphs.

We can construct a Jaeger's graph starting from any cubic 3-edge colourable graph. Indeed, consider a perfect matching M of G together with a bipartition of its vertex set in *Blue* and *Red* induced by a 3-edge colouring of G given by Theorem 2.3. If there are no mixed edge, we are done since G itself is a Jaeger's graph. Otherwise for any mixed edge apply the transformation depicted in figure 1 on the *Blue* vertices. Every such *Blue* vertex is transformed into a new triangle containing a new *Blue* edge while the mixed edge is transformed into a *Red* edge. The resulting graph is a Jaeger's graph .

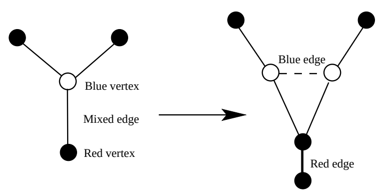

Figure 1: Triangle Extension

Let us recall that the *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest path joining them. The *square* G^2 of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever the distance $d(u, v)$ in G is at most 2.

Proposition 2.10 [5] *If G is a cubic graph such that G^2 is 4-chromatic then G is a Jaeger's graph .*

A cubic planar graph is a *multi-k-gon* [13] (with $3 \le k \le 5$) if all of its faces have length multiple of k.

Proposition 2.11 [5] *If G is a multi-k-gon with $k = 3, 4$ then G is a Jaeger's graph .*

Proof This result is a consequence of Proposition 2.10, since Jaeger [23] has shown that a multi-k-gon G with $k = 3, 4$ has a square G^2 which is 4-chromatic.

□

When G is a cubic graph having a 2-factor of C_4's, say $\mathcal{F}$, we consider the auxiliary graph G' defined as follows : every C_4 of $\mathcal{F}$ is replaced with its complementary graph which is a $2K_2$; G' is a two regular graph in which connected components are cycles.

Theorem 2.12 *Let G be a connected cubic graph having a 2-factor of squares, say $\mathcal{F}$ and let p be the number of cycles of G'. Then there are 2^{p-1} Jaeger's matchings which intersect $\mathcal{F}$.*

Proof We first prove that there are at most two types of Jaeger's matchings in G.

Claim *Let $M = M_B \cup M_R$ be a Jaeger's matching of G, if M intersects $\mathcal{F}$ then every C_4 of $\mathcal{F}$ contains an edge of M_B and an edge of M_R.*

Proof of Claim Recall that M_B and M_R are strong matchings. Without loss of generality we may assume that there is some edge say ab of some C_4 in $\mathcal{F}$, say $abcd$ which belongs to M_B. Since M is a perfect matching and M_B is a strong matching the vertices c and d must be the endpoint of some edge(s) of M_R. Since M_R is a strong matching we have $cd \in M_R$. Let $a'b'c'd'$ be another C_4 of $\mathcal{F}$ which is connected to $abcd$ by some edge say aa'. The edge aa' is not an edge of M (M is a matching), since a' must be an endpoint of an edge of M_R, M_R intersects $a'b'c'd'$. Consequently, G being connected we have that M_B and M_R intersect all cycles of $\mathcal{F}$. ■

It follows that a Jaeger's matching of G is either contained into $\mathcal{F}$ or disjoint from $\mathcal{F}$.
We now establish a correspondence between the orientations of the cycles of G' and the Jaeger's matchings of G which intersect $\mathcal{F}$.
Let us give an orientation of the cycles of G'. Going back now to G, each C_4 of $\mathcal{F}$ has an edge connected to two out-going edges and an edge connected to two in-going edges. Let M_B be the set of edges connected to two out-going edges over all the C_4's of $\mathcal{F}$ while M_R contains the edges connected to two in-going edges. It's an easy task to check that $M_B \cup M_R$ is a Jaeger's matching of G.
Conversely let us consider a Jaeger's matching $M = M_B \cup M_R$ of G which intersects $\mathcal{F}$. We know by the claim given above that each C_4 of $\mathcal{F}$ contains an edge of M_B and an edge of M_R. For any C_4 of $\mathcal{F}$ and for any vertex v of this C_4 we denote e_v the edge of $E(G)\backslash E(\mathcal{F})$ that is adjacent to v. We know that v is an endpoint of an edge in M_B or in M_R. We give an orientation to the edge e_v in such a way that e_v is an out-going edge (that is v is the origin) if and only if v is endpoint of an edge of M_B. Since every edge of $E(G)\backslash E(\mathcal{F})$ is connected to two C_4's of $\mathcal{F}$ those edges are oriented twice ; more precisely : when aa' is an edge connecting two cycles of $\mathcal{F}$, say $abcd$ and $a'b'c'd'$, if $aa' = e_a$ is an out-going edge for the cycle $abcd$ then $aa' = e_{a'}$ must be an in-going edge for $a'b'c'd'$ for otherwise M_B would no be a strong matching. Consequently the

given orientation of all edges e_v ($v \in V(G)$) extends to an orientation of the cycles of G'.

We have 2^p possible orientations of the cycles of G'. A given orientation of each cycle of G' and the opposite orientations of these cycles yield to the same partition of M, consequently, there are 2^{p-1} Jaeger's matchings intersecting the 2-factor $\mathcal{F}$ of G. This finishes the proof. □

By Theorem 2.12 every cubic graph having a 2-factor of squares has at least one Jaeger's matching. Hence we conclude this subsection with the following corollary.

Corollary 2.13 *A cubic graph having a 2-factor of squares is a Jaeger's graph.*

Furthermore, we can derive from Theorem 2.12 a simple linear time algorithm for finding a Jaeger's matching in a connected cubic graph which have a 2-factor of squares.

It can be noticed that every cubic graph with a perfect matching M can be transformed into a Jaeger's graph by using the transformation (*square extension*) depicted in figure 2 on each edge of M. Indeed, the resulting graph has a 2-factor of squares and we can apply Theorem 2.12

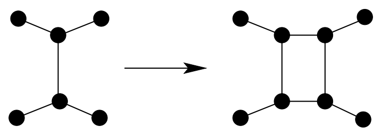

Figure 2: Square Extension

2.5 Construction

Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that a and d are internal vertices in L_B while b and c are internal vertices in L_R.
Let us consider the linear forest L_R and the edges of L_R incident to the vertices of $U = \{a, b, c, d\}$. We notice that, without loss of generality, there are six distinct cases (by exchanging the role of a for that of d or/and the role of b for that of c). See figure 3.

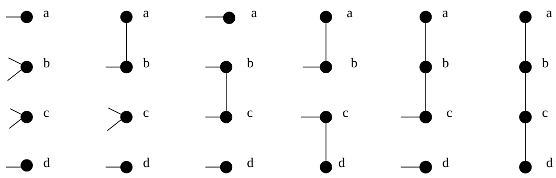

Figure 3: Six distinct cases

Analogous situations appear for the linear forest L_B (by exchanging $\{a, d\}$ for $\{b, c\}$).

Definition 2.14 Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that a and d are internal vertices in L_B while b and c are internal vertices in L_R. A (L, U)-*extension* of G is a cubic simple graph G' obtained from G in the following way.

> 1) The set U is splitted into two sets $U_R = \{a_R, b_R, c_R, d_R\}$ and $U_B = \{a_B, b_B, c_B, d_B\}$ (that is $V(G') = VG) \setminus U \cup (U_R \cup U_B))$.
>
> 2) For $x, y \in V(G) \setminus U$, if $xy \in E(G)$ then $xy \in E(G')$.
>
> 3) For $x \in V(G) \setminus U$ and $y \in U$ if $xy \in E(L_B)$ then $xy_B \in E(G')$ and if $xy \in E(L_R)$ then $xy_R \in E(G')$.
>
> 4) For $x, y \in U$ if $xy \in E(L_B)$ then $x_By_B \in E(G')$ and if $xy \in E(L_R)$ then $x_Ry_R \in E(G')$.
>
> 5) The remaining edges of G' are the edges of two paths of length 3 on the sets U_R and U_B, respectively, such that the obtained graph G' is cubic (see figures 4 and 5).

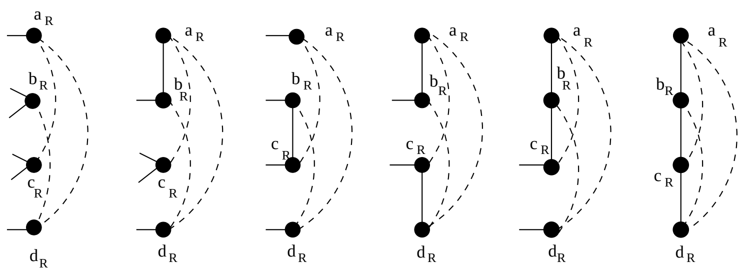

Figure 4: Addition of a path of length 3 on U_R

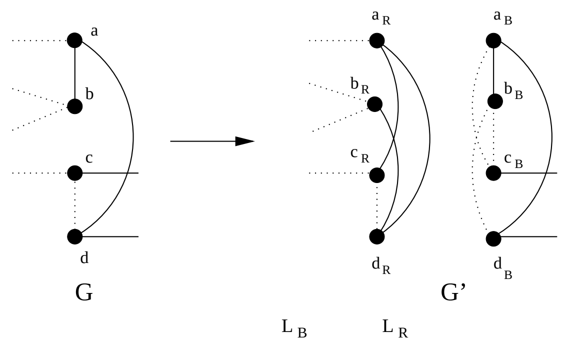

Figure 5: Example of (L, U)-extension

It is clear that the added path on U_R (respectively U_B) can be added to L_B (respectively L_R) in order to obtain a linear partition of G' each path of which has length 3.

Note that the graph G' is not uniquely defined, namely if the subgraph induced on U in L_R (respectively, in L_B) is a stable set or has exactly one edge connecting the vertices of degree 2 in L_R (respectively, in L_B).

So, by Theorem 2.8 we have the following Proposition.

Proposition 2.15 *Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of G each path of which has length 3. Let $U = \{a, b, c, d\}$ be a set of 4 vertices of G such that a and d are internal vertices in L_B while b and c are internal vertices in L_R. Then any $(L, U)-$extension of G on U is a Jaeger's graph.*

Definition 2.16 Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition for which every path has length 3. Assume that $P \in L_B$ and $Q \in L_R$ where $P = \{a_1, b_1, c_1, d_1\}$ and $Q = \{a_2, b_2, c_2, d_2\}$ are vertex disjoint paths in G. A *PQ-reduction* of G on P and Q is a cubic simple graph G' obtained from G by deleting the edges of P and Q and identifying the internal vertices of P with the end vertices of Q and the internal vertices of Q with the end vertices of P.

Note that the PQ-reduction of G has a linear partition each path of which has length 3. Hence we have the following.

Proposition 2.17 *Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of G every path of which has length 3. Assume that $P \in L_B$ and $Q \in L_R$ are vertex disjoint paths in G. Then the PQ-reduction of G on P and Q is a Jaeger's graph.* □

We get immediately from Propositions 2.15 and 2.17

Theorem 2.18 *Every Jaeger's graph on $n \geq 20$ vertices is obtained from a Jaeger's graph on 16 vertices by a sequence of $(L, U)-$extensions.*

Proof Assume that G is a Jaeger's graph on n vertices with $n \geq 20$. Let $L = (L_B, L_R)$ be a linear partition each of whose paths have length 3. Since each path of L is incident to at most 4 distinct paths, as soon as $n \geq 20$ we are sure to find a path in $P \in L_B$ and a path $Q \in L_R$ such that $V(P) \cap V(Q) = \emptyset$. By a PQ-reduction on these two paths we get a Jaeger's graph on $n-4$ vertices (Proposition 2.17). The proof is complete. □

3 Paths of length 3 in linear partitions.

In this section, we will develop some results about connection between linear partition and paths of length 3. In particular, we shall see that the unicoloured paths of length 3 play a special and important role.

Let G be a cubic graph with $|V(G)| = n \geq 4$ vertices and let $L = (L_B, L_R)$ be a linear partition of $E(G)$.

Definition 3.1 For $c \in \{B, R\}$, let $\mu(L_c)$ be the mean length of the paths of L_c, that is $\mu(L_c) = \frac{|E(L_c)|}{\omega(L_c)}$.

Lemma 3.2 *Let $L = (L_B, L_R)$ be a linear partition of a cubic graph G. Then*

$$\mu(L_B) \; = \; \frac{|E(L_B)|}{n - |E(L_B)|} \; = \; \frac{n - \omega(L_B)}{\omega(L_B)} \; = \; \frac{2\ \omega(L_R) + \omega(L_B)}{\omega(L_B)}$$

and

$$\mu(L_R) \; = \; \frac{|E(L_R)|}{n - |E(L_R)|} \; = \; \frac{n - \omega(L_R)}{\omega(L_R)} \; = \; \frac{2\ \omega(L_B) + \omega(L_R)}{\omega(L_R)}$$

Moreover, the following five conditions are equivalent:
a) $\omega(L_B) = \omega(L_R)$
b) $|E(L_B)| = |E(L_R)|$
c) $\mu(L_B) = \mu(L_R)$
d) $\mu(L_B) = 3$
e) $\mu(L_R) = 3$
Any of these five conditions implies $n \equiv 0$ $(mod\ 4)$.

Proof : Let $c \in \{B, R\}$. Since L_c is a spanning linear forest of G, we have

$$n \; - |E(L_c)| = \omega(L_c) \tag{1}$$

We also know that $\omega(L_B) + \omega(L_R) = \frac{n}{2}$. Hence

$$\mu(L_c) \; = \; \frac{|E(L_c)|}{n - |E(L_c)|} \; = \; \frac{n - \omega(L_c)}{\omega(L_c)} \tag{2}$$

and, by consequence,

$$\mu(L_B) \; = \; \frac{2\ \omega(L_R) + \omega(L_B)}{\omega(L_B)} \text{ and } \mu(L_R) \; = \; \frac{2\ \omega(L_B) + \omega(L_R)}{\omega(L_R)} \tag{3}$$

Using the equations (1) and (2) it is easy to see that conditions a), b) and c) are equivalent. Moreover, if $\omega(L_B) = \omega(L_R)$ then, by (3), we have $\omega(L_c) = 3$. If $\mu(L_B) = 3$ (or $\mu(L_R) = 3$) then (3) implies easily $\omega(L_B) = \omega(L_R)$. Hence d) and a) are equivalent.

By (2), $\mu(L_c) = 3$ implies $n = 4\ \omega(L_c) \ \equiv 0$ $(mod\ 4)$. This completes the proof of Lemma 3.2. □

Lemma 3.3 *Let $L = (L_B, L_R)$ be a linear partition of a cubic graph G. Then*

$$|\ \omega(L_B) \ - \ \omega(L_R)\ | \equiv \ \frac{n}{2} (mod\ 2)$$

Proof : For every pair $\{a, b\}$ of integers, it is easy to see that $|a - b| \equiv (a + b)(mod\ 2)$. Since $\omega(L_B) + \omega(L_R) = \frac{n}{2}$, we have $|\ \omega(L_B) - \omega(L_R)\ | \equiv \frac{n}{2}(mod\ 2)$. □

We recall that for $c \in \{B, R\}$, $l(L_c)$ is the length of a longest path in L_c.

Theorem 3.4 *Let $L = (L_B, L_R)$ be a linear partition of a cubic graph G. If $l(L_B) \leq 3$ then $\omega(L_B) \geq \omega(L_R)$. Moreover, if $l(F) = max(l(L_B), l(L_R))$ is at most 3 then $\omega(L_B) = \omega(L_R)$ (therefore $n \equiv 0\ (mod\ 4)$) and for $c \in B, R$, every maximal path in L_c has length 3 (that is G is Jaeger's graph).*

Proof : By Lemma 3.2 we have $\mu(L_B) = \frac{2\ \omega(L_R)+\omega(L_B)}{\omega(L_B)}$. If $l(L_B) \leq 3$ then $\mu(L_B) \leq 3$. Thus $\frac{2\ \omega(L_R)+\omega(L_B)}{\omega(L_B)} \leq 3$ and we obtain $\omega(L_B) \geq \omega(L_R)$.

If $max(l(L_B), l(L_R)) \leq 3$ then $\omega(L_B) \geq \omega(L_R)$ and $\omega(L_R) \geq \omega(L_B)$, that is $\omega(L_B) = \omega(L_R)$. By Lemma 3.2 we have $n \equiv 0\ (mod\ 4)$ and $\mu(L_B) = \mu(L_R) = 3$. Every unicoloured path P of (L_B, L_R) has length $l(P) \leq 3$ (by hypothesis). Thus, if there exists a path P_0 in L_B (respectively L_R) such that $l(P_0) < 3$ then we have $\mu(L_B) < 3$ (respectively $\mu(L_R) < 3$), a contradiction. Hence, every unicoloured path P of (L_B, L_R) has length exactly 3. □

Proposition 3.5 *Let $L = (L_B, L_R)$ be a linear partition of a cubic graph G such that $\omega(L_B) < \omega(L_R)$. Then there exists a linear partition (L'_B, L'_R) of G such that $\omega(L'_B) = \omega(L_B) + 1$ and $\omega(L'_R) = \omega(L_R) - 1$.*

Proof By Lemma 3.2 we have $\mu(L_B) > 3$. Then there exists a path $P = a_0a_1...a_k$ in L_B of length $k \geq 4$. For every i, with $1 \leq i \leq k-1$, a_i is an end vertex of a path in L_R. Since $k \geq 4$, there exists i, with $1 \leq i \leq k-1$, such that a_i and a_{i+1} are end vertices of two distinct paths Q_1 and Q_2 in L_R. Let us consider the edge a_ia_{i+1}. Let $L'_B = L_B \setminus \{a_ia_{i+1}\}$ and $L'_R = L_R \cup \{a_ia_{i+1}\}$. Then P is broken into two paths $P[a_0, a_i]$ and $P[a_{i+1}, a_k]$, and the paths Q_1 and Q_2 are connected by the edge a_ia_{i+1}. Clearly (L'_B, L'_R) is a linear partition such that $\omega(L'_B) = \omega(L_B) + 1$ and $\omega(L'_R) = \omega(L_R) - 1$. □

Proposition 3.6 *Let $L = (L_B, L_R)$ be a linear partition of a cubic graph G such that $\omega(L_B) \geq \omega(L_R)$. Let $\alpha = \omega(L_B) - \omega(L_R)$. For $c \in \{B, R\}$ and $j \in \{1, n-1\}$ let ${n_j}^c$ be the number of paths of length j in L_c. Then*

$$\sum_{j=1}^{n-1}(j-3){n_j}^B \ = \ -2\ \alpha \quad \text{and} \quad \sum_{j=1}^{n-1}(j-3){n_j}^R \ = \ 2\ \alpha\ .$$

Moreover, $l(L_B) \leq \frac{n}{2} - \alpha + 1$ and $l(L_R) \leq \frac{n}{2} + \alpha + 1$.

Proof By Lemma 3.2, $\mu(L_B) = \frac{2\ \omega(L_R)+\omega(L_B)}{\omega(L_B)}$ and $\mu(L_R) = \frac{2\ \omega(L_B)+\omega(L_R)}{\omega(L_R)}$. Then,

$$\mu(L_B) \ = \ 3 \ - \ \frac{2\ \alpha}{\omega(L_B)} \quad \text{and} \quad \mu(L_R) \ = \ 3 \ + \ \frac{2\ \alpha}{\omega(L_R)}\ .$$

For $c \in \{B, R\}$,

$$\sum_{j=1}^{n-1} j{n_j}^c \;=\; |E(L_c)| \;=\; \omega(L_c)\mu(L_c)\ .$$

Thus,

$$\sum_{j=1}^{n-1} j{n_j}^B \;=\; 3\ \omega(L_B) - 2\ \alpha \quad \text{and} \quad \sum_{j=1}^{n-1} j{n_j}^R \;=\; 3\ \omega(L_R) + 2\ \alpha$$

Since for $c \in \{B, R\}$

$$\sum_{j=1}^{n-1} {n_j}^c \;=\; \omega(L_c),$$

we have

$$\sum_{j=1}^{n-1} (j-3){n_j}^B \;=\; -2\ \alpha \quad \text{and} \quad \sum_{j=1}^{n-1} (j-3){n_j}^R \;=\; 2\ \alpha\ .$$

Then,

$$2{n_1}^B + {n_2}^B \;=\; 2\ \alpha + \sum_{j=4}^{n-1} (j-3){n_j}^B$$

and

$$2{n_1}^R + {n_2}^R \;=\; -2\ \alpha + \sum_{j=4}^{n-1} (j-3){n_j}^R\ .$$

Set $k = \frac{n}{2}$. Since $\omega(L_B) + \omega(L_R) = k$ and $\omega(L_B) - \omega(L_R) = \alpha$, we have $\omega(L_B) = \frac{k+\alpha}{2}$ and $\omega(L_R) = \frac{k-\alpha}{2}$. Let us suppose that for a given j, with $4 \leq j \leq n-1$, $n_j^B \geq 1$. Then, $2{n_1}^B + {n_2}^B \geq 2\ \alpha + (j-3)$. Since ${n_1}^B + {n_2}^B < \omega(L_B)$, we have ${n_1}^B > 2\ \alpha + j - 3 - \frac{(k+\alpha)}{2}$. Thus, $2\ \alpha + j - 3 - \frac{(k+\alpha)}{2} < {n_1}^B < \omega(L_B) = \frac{k+\alpha}{2}$. Then, $j < k - \alpha + 2$. We have proved that if $j \geq k - \alpha + 2$ then there is no path of length j in L_B, that is to say $l(L_B) \leq k - \alpha + 1$. Analogously, if for a given j, with $4 \leq j \leq n-1$, $n_j^R \geq 1$ we can prove that $j < k+\alpha+2$ and $l(L_R) \leq k+\alpha+1$. $\square$

Now, let us recall a result of Aldred, Jackson, Lou and Saito:

Theorem 3.7 [2] *Every cubic graph G has a linear partition $L = (L_B, L_R)$ such that*

(1) $\omega(L_B) = \omega(L_R)$ if and only if $n \equiv 0\ (mod\ 4)$

(2) $\omega(L_B) = \omega(L_R) + 1$ if and only if $n \equiv 2\ (mod\ 4)$

Proof : Let $L = (L_B, L_R)$ be a linear partition of G. We know that $\omega(L_B) + \omega(L_R) = \frac{n}{2}$. By Lemma 3.3, $|\ \omega(L_B) - \omega(L_R)\ | \equiv \alpha\ (mod\ 2)$ with

$$\alpha = \begin{cases} 0 & \text{if } n \equiv 0\ (mod\ 4) \\ 1 & \text{if } n \equiv 2\ (mod\ 4) \end{cases}$$

Without loss of generality, suppose that $\omega(L_B) < \omega(L_R)$. Then $\omega(L_R) - \omega(L_B) = \alpha + 2\ r$ with $r \geq 0$. If $r = 0$ then the theorem follows. Suppose that $r \geq 1$. Let (L'_B, L'_R) be a linear partition obtained by the effective procedure given in the proof of Proposition 3.5. We have $\omega(L'_B) = \omega(L_B) + 1$ and $\omega(L'_R) = \omega(L_R) - 1$. Hence $\omega(L'_R) - \omega(L'_B) = \alpha + 2(r - 1)$. Then, by induction and Proposition 3.5, we obtain a sequence $\{({L_B}^{(j)}, {L_R}^{(j)})\}_{1 \leq j \leq r}$ of linear partitions of G with ${L_c}^{(1)} = L'_c$ and such that for any integer j in $\{1, r\}$, $\omega({L_R}^{(j)}) - \omega({L_B}^{(j)}) = \alpha + 2(r - j)$. Therefore $\omega({L_R}^{(r)}) - \omega({L_B}^{(r)}) = \alpha$ and the theorem follows. □

4 Semi-odd and odd linear partitions

There are relationships between semi-odd linear partitions and perfect matchings, and between odd linear partitions and 3-edge colourability.

4.1 Semi-odd linear partitions and perfect matchings

Recall that a semi-odd linear partition is a linear partition where at least one forest is odd.

Theorem 4.1 [16] *Let G be a cubic graph having a perfect matching M. Then there exists a set $F \subseteq E(G) - M$ intersecting each cycle of the 2-factor $G - M$ such that $F + M$ is an odd linear forest.*

Theorem 4.2 [16] *A cubic graph has a perfect matching if and only if it has a semi-odd linear partition.*

For any cubic graph G having a perfect matching we denote by $\rho(G)$ the minimum number of even maximal paths appearing in a semi-odd linear partition. If $\rho(L)$ denotes the number of even maximal paths of a semi-odd linear partition $L = (L_B, L_R)$, then $\rho(G) = Min\{\rho(L) |\ L \text{ is a semi} - \text{odd linear partition of } G\}$. For any cubic graph G having a 2-factor we denote by $o(G)$ the minimum number of odd cycles appearing in a 2-factor of G (we note that $o(G)$ is an even number).

Theorem 4.3 [16] *Let G be a cubic graph having a 2-factor (or, equivalently, a perfect matching M). Then $\rho(G) = o(G)$.*

Corollary 4.4 *(see* [3]*) Let G be a cubic graph having a perfect matching. Then the follwing properties are equivalent:*

1. *$\rho(G) = 0$*
2. *G is 3-edge colourable (that is $\chi'(G) = 3$).*
3. *G can be factored into two odd linear forests*

4.2 Odd linear partitions and 3-edge colourability

Let G be a cubic graph. Assume that $L = (L_B, L_R)$ is a linear partition of its edge set. By colouring alternately the edges of the maximal paths in L_B with α and γ and those of L_R with β and δ, we get a 4-edge colouring. Aldred and Wormald proved :

Theorem 4.5 [3] *Let G be a cubic graph. Then G can be factored into two odd linear forests if and only if $\chi'(G) = 3$.*

Assume that G is a cubic 3-edge colourable graph and consider a 3-edge colouring of G. Let α and β be any two distinct colours. The subset of $E(G)$ coloured with α or with β induces an even 2-factor (spanning subgraph components of which are even cycles). In the following the 2-factor induced by any two distinct colours α and β will be denoted by $\Phi(\alpha, \beta)$. Any cycle of $\Phi(\alpha, \beta)$ is said to be an $\alpha\beta$-cycle. Since any edge of a 3-edge colourable cubic graph belongs to a 2-factor, it is clear that the connected components of G induce 2-connected subgraphs.

Definition 4.6 Let α and β be any two distinct colours of $E(G)$. In the following $SM_G(\alpha, \beta)$ will denote a strong matching of G intersecting every $\alpha\beta$-cycle (when such a strong matching exists).

Theorem 4.7 [16] *Let G be a 3-edge coloured cubic graph and let α and β be any two distinct colours of $E(G)$. Then there exists a strong matching $SM_G(\alpha, \beta)$ intersecting every cycle of $\Phi(\alpha, \beta)$.*

Corollary 4.8 [16] *Let G be a cubic graph. Then G can be factored into two odd linear forests $L = (L_B, L_R)$ such that*

i) Each path in L_B has odd length at most 3

ii) Each path in L_R has odd length at least 3.

if and only if G is 3-edge colourable.

Proof Assume that G has an odd linear partition $L = (L_B, L_R)$ with these properties. As in Theorem 4.5 we get immediately a 3-edge colouring.
Conversely, let us consider a 3-edge colouring of G. Let α and β be any two distinct colours of $E(G)$ and let γ be the third colour. Let M_γ be the perfect matching of the γ-coloured edges. Let $SM_G(\alpha, \beta)$ be a minimal strong matching intersecting each cycle of $\Phi(\alpha, \beta)$. Then, $L_B = M_\gamma + SM_G(\alpha, \beta)$ is a set of odd paths of length at most 3, while $L_R = \Phi(\alpha, \beta) \setminus SM_G(\alpha, \beta)$ is a set of odd paths of length at least 3 (recall that, G being simple, every bicoloured cycle has length at least 4). Hence, (L_B, L_R) is an odd linear partition satisfying our conditions.

□

We derive from Theorem 4.7 a result on unicoloured transversals of the 2-factors induced by any 3 edge-colouring of cubic graph with chromatic index 3.

Theorem 4.9 [16] *Let G be a cubic 3-edge colourable graph and let Φ be a 3-edge colouring of G. Let α and β be any two distinct colours of Φ and let γ be the third colour. Then there exists a set F_α of α-edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that the set F_α together with the γ-edges has no cycle.*

Remark 4.10 It is possible to derive a linear time algorithm for the construction of the unicoloured transversal F_α of Theorem 4.9 once a 3-edge colouring and the strong matching described in Theorem 4.7 are given (see [16]).

4.3 Open problems

In the introduction of this paper we recalled that for any cubic graph G, $la_1(G) = 3$ or 4, $la_2(G) = 3$, for $k \in \{3, 4\}$ $2 \leq la_k(G) \leq 3$ and (by Thomassen [28]) for $k \in [5, n-1]$ $la_k(G) = la(G) = 2$.

Thomassen's result is the best possible since, in view of $la_4(K_{3,3}) = 3$ and $la_4(PR_3) = 3$, 5 cannot be replaced by 4. As far as it is known, these two graphs are the only exceptions and it could be true that $la_4(G) = 4$ with the exception of $K_{3,3}$ and PR_3. Jackson and Wormald ([24]) propose thus as an open problem:

Problem 4.11 Is it true that if G is a cubic simple graph with at least 8 vertices then $la_4(G) = 2$?

The answer to this problem when considering even the restricted class of planar cubic graph could be of some interest.

In Theorem 2.9 we have seen a class of cubic 3-edge colourable graphs, containing Jaeger's graphs, where each graph can be provided with an odd linear partition (into two forests) where each path has length at most 7. Aldred and Wormald [3] proved that any cubic 3-edge colourable graph has a linear partition (not necessarily odd) each of whose paths have length at most 7. Since we know, by Thomassen's result [28], that any cubic graph has a linear partition in which every path has length at most 5, we propose as an open problem to find an analogous universal bound for odd linear partition in cubic 3-edge colourable graphs.

Problem 4.12 Is it true that if G is a cubic 3-edge colourable graph then there is an odd linear partition in which every path has length at most 5?

5 On isomorphic linear partitions

For any cubic graph on $n \equiv 0 \ (mod\ 4)$ vertices, we have seen in Theorem 3.7 that we can find a linear partition where $\omega(L_B) = \omega(L_R)$. Moreover, by Proposition 3.6, we can say that, from a statistical point of view, these two linear forests are identical. In fact, a conjecture of Wormald [30] goes further in that direction.

Conjecture 5.1 [30] *Let G be a cubic graph with $|E(G)| \equiv 0 \ (mod\ 2)$ (or equivalently $|V(G)| \equiv 0 \ (mod\ 4)$). Then there exists a linear partition $L = (L_B, L_R)$ of $E(G)$ such that L_B and L_R are isomorphic linear forests.*

Remark 5.2 Theorem 2.8 implies that Conjecture 5.1 is true for Jaeger's graphs. Up to our knowledge, it is even the only known class with this property.

Our purpose, in that section, is to give some new results concerning this conjecture.

5.1 On cubic 3-edge colourable graphs

Assume that G is a cubic 3-edge colourable graph and let $M = M_B + M_R + M'$ be an associated partition (see Definition 2.5). Let us recall that a Jaeger's graph has an associated partition $M = M_B + M_R$ (M' is empty). One may ask what happens when $\mid M' \mid$ is bounded. A partial answer is given:

Theorem 5.3 [15] *Let G be a cubic 3-edge colourable graph on $n \equiv 0$ (4) vertices. Assume that we can find a 3-edge colouring with an associated partition $M = M_B + M_R + M'$ where M' has exactly two edges. Then there exists a linear partition $L = (L_B, L_R)$ of $E(G)$ such that L_B and L_R are isomorphic linear forests.*

Definition 5.4 A linear partition $L = (L_B, L_R)$ of a cubic graph G such that L_B and L_R are isomorphic linear forests is said to be an *isomorphic linear partition.*

We shall see now that cubic 3-edge colourable graphs with a $2-$factor of triangles can be provided with an isomorphic odd linear partition.

Theorem 5.5 [15] *Let G be a cubic 3-edge colourable graph on $n \equiv 0$ (4) vertices and let $M = M_B + M_R + M'$ be an associated partition. Assume that for any two edges e and e' in M' the shortest alternating path (that is a path $v = v_0v_1v_2 \ldots v_{2k+1} = w$ such that any edge v_iv_{i+1}, where i is odd, is an edge of M) joining these two edges have length at least 5. Then G has an odd isomorphic linear partition.*

Up to our knowledge, 3-edge colourable cubic graphs with a $2-$factor of triangles are the only examples satisfying 5.5.

Corollary 5.6 [15] *Let G be a cubic 3-edge colourable graph on $n \equiv 0$ (4) having a $2-$factor of triangles. Then G has an odd isomorphic linear partition.*

Proof Assume that G is 3-edge coloured and let $M = M_B + M_R + M'$ an associated partition. It is an easy matter to see that each triangle contains an edge of M_B or M_R while exactly one edge connecting this triangle to another one is also in M. Hence the three edges of each triangle are affected in the associated linear construction either to L_B or to L_R. The edges of M' are edges connecting some triangles of our $2-$factor (each triangle being incident to at most one edge of M'). If M' is empty, G is a Jaeger's graph and we are done. M' being even, let $xy, x'y'$ two distinct edges of M', we want to show that their alternating distance is at least 5.
Assume that x is contained in the triangle xuv and y in ywt while x' is contained in $x'u'v'$ and y' in $y'w't'$. A shortest alternating path joining xy to $x'y'$ begins either with xuv or xvu or ywt or else ytw. In the same way, it must ends with

$v'u'x'$ or $u'v'x'$ or $t'w't'$ or $w't'y'$. Since each triangle is incident to at most one edge of M', a shortest alternating path has length at least 5. The conclusion follows from theorem 5.5. □

We have seen in Theorem 2.12 that a cubic graph G having a 2-factor of squares is a Jaeger's graph and, hence, can be provided with an isomorphic linear partition. As a step towards conjecture 5.1, it could be interesting to generalize these results when considering $k-$uniform 2-factors (each cycle has length k for a fixed $k \geq 5$).

5.2 Graphs with strong chromatic index 5

A *strong edge colouring* of a graph G is a partition of its edge set into strong matchings. Let $\chi_S(G)$ (*strong chromatic index*) denote the minimum integer k for which $E(G)$ can be partitioned into k strong matchings of G. This notion was introduced in [13] and [14] while [11] is the usual reference for the origin of this problem. When dealing with cubic graphs, we have immediately that $\chi_S(G) \geq 5$. We know that $\chi_S(G) \leq 10$ (see [4] and [22]) for cubic graphs in general and $\chi_S(G) \leq 9$ (see [27]) when considering cubic bipartite graphs (answering thus positively to conjectures appearing in [14] and [12]).
The class of cubic graphs satisfying $\chi_S(G) = 5$ (as Petersen's graph, Dodecahedron and the graphs associated to C_{60} the molecule of the well known fulleren) is of particular interest. A simple counting argument leads to $|V(G)| \equiv 0 \ (10)$. By the way, this implies that $\chi_S(G) \geq 6$ when $|V(G)| \not\equiv 0 \ (10)$ which gives us easy counterexamples to a conjecture in [12] saying that $\chi_S(G) = 5$ when G is a cubic bipartite graph with girth sufficiently large.

We have also the following structural result:

Theorem 5.7 [14, 15] *Let G be a cubic graph with $\chi_S(G) = 5$. Then for every strong 5-edge colouring of G the spanning subgraph of G obtained in considering three any colours is union of an induced subgraph $\mathcal{K}$ of G made of k disjoint $K_{1,3}$ and of an induced subgraph $\mathcal{C}$ union of disjoint cycles without chord of length multiple of 6. The sum of the lengths of these cycles is equal to two times the number of pendent vertices of $\mathcal{K}$, that is $6k$. Moreover, for any two positive integers p and q such that $p+q = k$, G has a linear partition $L = (L_B, L_R)$ such that*

- *L_B is a set of p paths of length 6, $p + 2q$ paths of length 2 and q paths of length 3*
- *L_R is a set of q paths of length 6, $q + 2p$ paths of length 2 and p paths of length 3*

By Remark 5.2 and Proposition 2.11 we know that multi-3-gons and multi-4-gons satisfy conjecture 5.1. Multi-5-gons are not Jaeger's graphs in general, however we can show that they do have an isomorphic linear partition.

Corollary 5.8 *Let G be a multi-5-gon. Then G can be partitioned into two isomorphic linear forests.*

Proof In [13], it is proved that the strong chromatic index of a multi-5-gon is 5 and its number of vertices is multiple of 20. In that case the number k of $K_{1,3}$ is even. From theorem 5.7 we consider $p = q = \frac{k}{2}$ and we get the result.
□

6 Compatible linear partitions

Let $L = (L_B, L_R)$ be a linear partition of G. For each vertex v we can define $e_L(v)$ as the edge incident to v which is an end edge of a maximal path in L_B or L_R. We shall say that two linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are *compatible* whenever $e_L(v) \neq e_{L'}(v)$ for each vertex v. The qualifying adjective "compatible" refers to the notion of compatible Euler's tours introduced by Kotzig [25] (see Bondy [6] for an introduction to this question).

6.1 Compatible partitions and associated linear construction

In view of the role played by Jaeger's graphs, it is not surprising to see that these graphs can be provided with two compatible linear partitions. In fact, we have a more general result.

Theorem 6.1 *Let G be a cubic 3-edge colourable graph with an associated partition $M = M_B + M_R + M'$. Assume that, we can colour the edges of M' in Blue and Red in such a way that, for each associated linear construction, the whole colouring of $E(G)$ so obtained is a linear partition. Then G has two compatible linear partitions.*

Proof An associated linear construction is obtained in fixing an arbitrary orientation to the cycles of $G \setminus M$. To each vertex v of $V(G)$ we associate an edge $o(v)$ of $E(G) \setminus M$ such that v is the origin of $o(v)$ with respect to the chosen orientation of the cycle through v. We colour $o(v)$ in *Blue* or *Red* accordingly to the colour of v.
We give a colour to each edge of M', accordingly to some rule depending on the class of graphs we are studying. When the edge of M' incident to some vertex v is coloured in *Blue* or *Red*, v is transformed into a vertex of degree 2 in one of the two colours (say *Blue*) and a vertex of degree 1 in the other (*Red*). Since the whole colouring of $E(G)$ leads to a linear partition $L = (L_B, L_R)$, the edge incident to v of this last colour is hence the edge $e_L(v)$.
On each cycle of $G \setminus M$ we can give now the opposite orientation. In the associated linear construction, the colours of the edges of the cycles incident to each vertex are exchanged. Since, with the same colouring of M', we have supposed that the whole colouring of $E(G)$ leads to a linear partition $L' = (L'_B, L'_R)$, the above vertex v remains a vertex of degree 2 in the *Blue* colour. Hence the *Red* edge incident to v $e_{L'}(v)$ is distinct from $e_L(v)$.

Since $e_L(v) \neq e_{L'}(v)$ for each vertex v, the two linear partitions L and L' so obtained are compatible. $\square$

Corollary 6.2 *Let G be a Jaeger's graph then G has two compatible linear partitions in which every path has length* 3.

Proof In that case, we have a perfect matching $M = M_B + M_R$ and any associated linear construction is a linear partition in which every path has length 3. $\square$

In the same way we have,

Corollary 6.3 *Let G be cubic* 3*-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that M' can be partitioned into two strong matchings M'_B and M'_R. G has two compatible linear partitions in which every path has length* $1, 3, 5$ *or* 7.

Proof In proof of Theorem 2.9 we have coloured the edges of M'_B in *Blue* and those of M'_R in *Red* and we have shown that the associated linear construction together with this colouring of M' leads to a linear partition with each path of length $1, 3, 5$ or 7. Applying Theorem 6.1 above leads to the result. $\square$

It can be pointed out that the two compatible linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ obtained in Corollaries 6.2 and 5.6 are such that the four linear forests L_B, L_R, L'_B and L'_R are isomorphic. This is not necessarily true in Corollary 6.3.

Corollary 6.2 lead the first author in 1991 to conjecture:

Conjecture 6.4 *Every cubic graph has two compatible linear partitions.*

6.2 Compatible partitions in hamiltonian cubic graphs

As a corollary of Theorem 4.7, we shall show now that cubic hamiltonian graphs satisfy Conjecture 6.4. In fact, a stronger result is obtained since the two compatible partitions are odd.

Theorem 6.5 *Let G be a cubic hamiltonian graph, then G has two compatible odd linear partitions.*

Proof Let $C = a_0, a_1 \dots a_{n-1}$ be a hamiltonian cycle of G. This hamiltonian cycle induces a 3-edge colouring $\Phi : \; E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us colour every edge $a_i a_{i+1}$ of C with α when $i \equiv 0(2)$ and with β otherwise, while the remaining perfect matching is coloured with γ. From Theorem 4.7 we know that there is a strong matching F intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$. We choose F minimal for the inclusion (that is F intersects each cycle of $\Phi(\beta, \gamma)$ in exactly one edge). Let M_α be the set of α-coloured edges. Since F is a strong matching intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$, we can construct an odd linear partition $L' = (L'_B, L'_R)$:

$$L'_R = M_\alpha \cup F$$

$$L'_B = E(G) - L'_R$$

For each vertex v, the edge $e_{L'}(v)$ is coloured with α excepted when v is an end vertex of an edge of F. In that case $e_{L'}(v)$ coloured with γ when v is an end vertex of the edge of $F \cap C$ and with β when v is an end vertex of an edge of $F \setminus C$.

Case 1 : *F contains some edges of C*
Hence the edges of $F \cap C$ are coloured with β. $F \cap C$ is a strong matching intersecting the 2-factor made of the unique cycle C leading to the following odd linear partition $L = (L_B, L_R)$:

$$L_B = C - F$$

$$L_R = E(G) - L_B$$

For each vertex v, $e_L(v)$ is coloured with γ excepted when v is an end vertex of the edge of $F \cap C$. In that case we have $e_L(v)$ coloured with β.
We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex v, since the colours of these edges are distinct. Hence, the two odd partitions L and L' are compatible.

Case 2 : *Each edge of F is coloured with γ and there is an edge $a_i a_{i+1}$ (i odd) of C coloured with β which is not incident to an edge of F.*
Let $L = (L_B, L_R)$ be the following odd linear partition:

$$L_B = C - a_i a_{i+1}$$

$$L_R = E(G) - L_B$$

For each vertex v, $e_L(v)$ is coloured with γ excepted when v is a_i or a_{i+1}. In that case $e_L(v)$ is coloured with α.
For each vertex v, $e_{L'}(v)$ is coloured with α unless when v is an end vertex of an edge of F. In that case $e_{L'}(v)$ is coloured with β when v is an end vertex of an edge of F.
We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex v, since the colours of these edges are distinct. Hence, the two odd partitions L and L' are compatible.

Case 3 : *Each edge of F is coloured with γ and each edge of C coloured with β is incident to an edge of F.*
Without loss of generality assume that a_2 is incident to F. Edge a_3a_4 being incident to F, we must have a_4 incident to F. Going through C, we get that $a_2, a_4, \dots a_{2k}, \dots a_0$ must be incident to F. The remaining edges coloured with γ are edges joining vertices of C with odd index. It is an easy task to see that this set K of edges is a strong matching. The perfect matching coloured with γ is the union of two strong disjoint matchings F and K with the same size. Hence G is a Jaeger's graph . Theorem 6.2 implies that we have two compatible odd linear partitions. $\square$

6.3 Compatible odd linear partitions

The results of Theorem 6.5 and Theorem 6.1 leads us to a strengthening of Conjecture 6.4.

Conjecture 6.6 *Let G be a cubic 3-edge colourable graph then we can find two compatible odd linear partitions.*

As a partial result we have:

Theorem 6.7 *Let G be a cubic 3-edge colourable graph then we can find three odd linear partitions L, L' and L'' such that for each vertex v*

$$|\{e_L(v), e_{L'}(v), e_{L''}(v)\}| \geq 2$$

Proof Let us consider a 3-edge colouring $\Phi : E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us denote by M_γ the perfect matching consisting of the γ-coloured edges. Theorem 4.9 implies that there exists a set F_α of α-coloured edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that $F_\alpha \cup M_\gamma$ is acyclic. In that way, we obtain an odd linear partition $L = (L_1, L_2)$

$$L_1 = F_\alpha \cup M_\gamma$$

$$L_2 = E(G) - L_1$$

For each vertex v we have $e_L(v)$ coloured with γ excepted for the vertices which are the end vertices of an edge of F_α. In that case $e_L(v)$ is coloured with β.

Let us consider the perfect matching M_β, the bicoloured cycles of $\Phi(\alpha, \gamma)$ and a matching F_γ obtained by Theorem 4.9. Hence, we get an odd linear partition $L' = (L'_1, L'_2)$ such that for every vertex v the edge $e_{L'}(v)$ is coloured either with β (when the vertex v is an end vertex of an edge of F_γ) or with α. Finally we obtain a third odd linear partition $L'' = (L''_1, L''_2)$ from the perfect matching M_α, the bicoloured cycles of $\Phi(\beta, \gamma)$ and a matching F_β, such that the edges $e_{L''}(v)$ are coloured α or γ (γ when the considered vertices are the end vertices of an edge of F_β).

The three sets F_α, F_β and F_γ being obviously pairwise disjoint, it is a routine matter to see that for each vertex v two edges in $\{e_L(v), e_{L'}(v), e_{L''}(v)\}$, at least, have distinct colours. These two edges are thus distinct and we get the result.

□

7 Complexity

The following result of Akiyama et al.[1] can be also obtained by an algorithmic way (see [9]).

Theorem 7.1 *Let G be a cubic multigraph on $n \geq 4$ vertices each of whose components are distinct from Θ (the cubic multigraph on two vertices and three parallel edges). One can construct a linear partition $L = (L_B, L_R)$ of G in time-complexity $O(n^2)$.*

Let us recall that for any cubic graph G, $la_1(G) = 3$ or 4 (note that it is an NP-complete problem for a given cubic graph G to decide if G is 3-edge colourable [21]), $la_2(G) = 3$, for $k \in \{3, 4\}$ $2 \leq la_k(G) \leq 3$ and for $k \in [5, n-1]$ $la_k(G) = la(G) = 2$ (Thomassen [28]). Since it is an NP-complete problem to decide whether a cubic graph is a Jaeger's graph (Schaefer [26]), it is easy to prove that deciding whether $la_3(G) = 2$ or 3 is an NP-complete problem (see [5]). As previously said (see Subsection 4.3, Problem 4.11), determining whether $la_4(G) = 2$ or 3 is an open problem.

7.1 An NP-complete problem

By Theorem 2.8, for a cubic graph G on $n = 2k$ vertices (k even) to decide whether G has a linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = 3$ is an NP-complete problem. If L_B and L_R are isomorphic forests then, by Proposition 3.6, for $c \in \{B, R\}$, $3 \leq l(L_c) \leq k + 1$. This suggests the following decision problem:

> **Instance :** A cubic graph G of order $n = 2k$ and an integer q such that $3 \leq q \leq k + 1$.
>
> **Question :** Does there exists a linear partition (L_B, L_R) of G such that $l(L_B) = l(L_R) = q$?

complexity of which is not known in general. However we show below that this problem is NP-complete when $q = k + 1$.

Lemma 7.2 *Let G be a cubic graph of order $n = 2k$. Let P and Q be two edge-disjoint paths of G such that $l(P) = l(Q) = k + 1$, then P and Q have exactly 4 common vertices and $V(P) \cup V(Q) = V(G)$.*

Proof Set $p = |V(P) \cap V(Q)|$. Since G is a cubic graph, it is clear that $0 \leq p \leq 4$. We have $|V(Q)| \leq |V(G)| - |V(P)| + |V(P) \cap V(Q)|$. Thus, $|V(Q)| \leq 2k - (k + 2) + p$, that is $p \geq |V(Q)| - k + 2 = 4$. Hence, $p = 4$ and $V(P) \cup V(Q) = V(G)$ as claimed.

□

Lemma 7.3 *Let G be a cubic graph on $n = 2k$ vertices with a linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = k + 1$. Then L_B and L_R are two isomorphic linear forests. Each of them contains exactly one path of length $k + 1$, k is even, and the set of the remaining paths is a matching of $\frac{k-2}{2}$ edges.*

Proof Let us suppose that G has a linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = k + 1$ (with $n = 2k$). By Lemma 7.2, L_B has a unique longest path P_0 and L_R has a unique longest path Q_0, with $l(P_0) = l(Q_0) = k + 1$, $V(G) = V(P_0) \cup V(Q_0)$ and $|V(P_0) \cap V(Q_0)| = 4$. Then every path of L_B distinct from P_0 is an edge end vertices of which belong to $V(Q_0)$ and every vertex of $Q_0 \setminus P_0$ is end vertex of a path of L_B. Since $|V(Q_0)| = k + 2$, $|V(P_0) \cap V(Q_0)| = 4$ and L_B is a spanning forest, k is even and $L_B \setminus E(P_0)$ is a matching on $\frac{k-2}{2}$ edges. By symmetry, $L_R \setminus E(Q_0)$ is also a matching on $\frac{k-2}{2}$ edges. Hence L_B and L_R are isomorphic forests. □

Definition 7.4 Let G and G' be two disjoint cubic graphs. Assume that $P = (a, b, c, d)$ is a path of length 3 in G and $Q = (b', d', a', c')$ is a path of length 3 in G'. We get a new cubic graph $H = PQ(G, G')$ (*PQ-construction* for short) by deleting the edges ab, bc, cd of P and $b'd'$, $d'a'$, $a'c'$ of Q and identifying vertices a and a', b and b', c and c', d and d' (see figure 6).

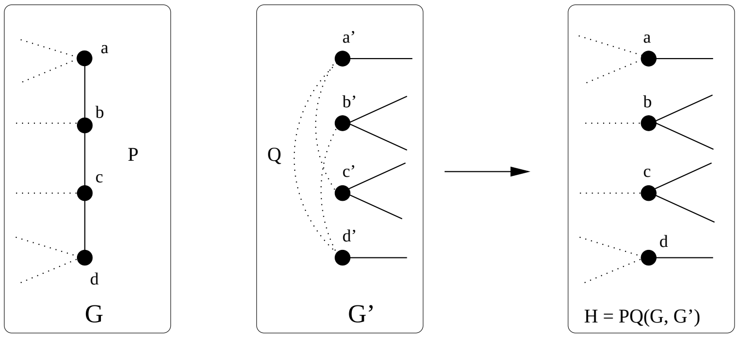

Figure 6: $PQ - Construction$

A natural question arises: how to obtain a linear partition of $H = PQ(G, G')$ from linear partitions of G and G'?

Proposition 7.5 *Let G and G' be two disjoint cubic graphs. Assume that G has a linear partition $L = (L_B, L_R)$ containing a path P of length 3 in L_R and that G' has a linear partition $L' = (L'_B, L'_R)$ containing a path Q of length 3 in L'_B. Then $L'' = (L''_B, L''_R)$, with*

$$L''_B = L_B + (L'_B - Q)$$
$$L''_R = (L_R - P) + L'_R$$

is a linear partition of $H = PQ(G, G')$.

Proof Assume that $P = (a, b, c, d)$ and $Q = (b', d', a', c')$. Since $P \in L_R$ the edges of G incident to the vertices of P are thus in L_B. In the same way, the edges of G' incident to the vertices of Q are in L'_R. Hence $L''_R = (L_R - P) + L'_R$ is a linear forest of H as well as $L''_B = L_B + (L'_B - Q)$. $\square$

Under the same hypotheses and notations of Proposition 7.5 we have the following.

Corollary 7.6 *If $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are isomorphic linear partitions (respectively isomorphic odd linear partitions) of G and G' then $L'' = (L''_B, L''_R)$ is an isomorphic linear partition (respectively isomorphic odd linear partition) of $H = PQ(G, G')$.*

Proof Since $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are isomorphic linear forests, for any j , with $1 \leq j \leq n-1$, L_B and L_R (L'_B and L'_R, respectively) have the same number of maximal paths of length j. Thus, for $j \neq 3$, L''_B and L''_R have the same number of maximal paths of length j. Since we delete a path of length

3 in L_R and a path of length 3 in L'_B, we get also the same number of maximal paths of length 3 in L''_B and L''_R. Thus, $L'' = (L''_B, L''_R)$ is an isomorphic linear partition (isomorphic odd linear partition, if L and L' are odd). □

When L and L' are odd linear partitions, we shall refer to the PQ-construction described in Proposition 7.5 as an *odd PQ-construction.*

When G is a cubic hamiltonian graph on $k+2$ vertices, we have a natural odd linear partition $L = (L_B, L_R)$ with L_B as a hamiltonian path obtained from a hamiltonian cycle with one edge deleted (any edge) and L_R as the remaining edges. In that particular linear partition, we have exactly one path of length 3 in L_R (the other paths have length 1).

Proposition 7.7 *Let G and G' be two disjoint cubic hamiltonian graphs on $k+2$ vertices. Let $L = (L_B, L_R)$ be an odd linear partition of G where L_B is a hamiltonian path and $P = (a, b, c, d)$ is the unique path of length 3 in L_R. In the same way let us consider an odd linear partition $L' = (L'_B, L'_R)$ of G' where L'_R is a hamiltonian path and $Q = (b', d', c', a')$ is the unique path of length 3 in L'_B. Then $H = PQ(G, G')$ is a cubic graph on $2k$ vertices with an isomorphic odd linear partition $L'' = (L''_B, L''_R)$ such that $l(L''_B) = l(L''_R) = k+1$*

Proof From Proposition 7.5 we know that $L'' = (L''_R, L''_B)$ (with $L''_R = (L_R - P) + L'_R$ and $L''_B = L_B + (L'_B - Q)$) is an odd linear partition of H. L''_R contains one long path, namely the hamiltonian path of G' which was in L'_R as well as L''_B contains the hamiltonian path of G which was in L_B. Since these paths have length $k+1$, we have $l(L''_B) = l(L''_R) = k+1$, as claimed. □

Proposition 7.8 *Let G be a cubic graph on $2k$ vertices with an odd linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = k+1$. Then G can be obtained from two cubic hamiltonian graphs on $k+2$ vertices by an odd PQ-construction.*

Proof By Lemma 7.3, $L = (L_B, L_R)$ is an odd isomorphic linear partition with exactly one path P_B of length $k+1$ and $\frac{k-2}{2}$ edges in L_B (one path P_R of length $k+1$ and $\frac{k-2}{2}$ edges in L_R). Assume that the end vertices of P_B are x_B and y_B (x_R and y_R are the end vertices of P_R). Let G_B be the subgraph of G, edges of which are the edges of P_B and the paths of length 1 in L_R (G_R being defined analogously with P_R). In G_B, x_B and y_B have degree 1, x_R and y_R (as internal vertices of P_B) have degree 2 and the remaining vertices have degree 3. The vertices x_B and y_B are not adjacent in G_B (otherwise P_B would have no internal vertex). Since $d(x_B) = 1$ in G_B and $x_R \neq y_R$, x_B is adjacent to at most one vertex in $\{x_R, y_R\}$. In the same way, y_B is adjacent in G_B to at most one vertex in $\{x_R, y_R\}$, x_R (y_R, respectively) is adjacent in G_R to at most one vertex in $\{x_B, y_B\}$. We can thus complete G_B in order to obtain a cubic hamiltonian graph G_1 by adding the edge x_By_B and the edges $\{x_Bx_R, y_By_R\}$ if $x_Bx_R \notin E(G_B)$ or the edges $\{x_By_R, y_Bx_R\}$ forming thus a path P of length 3 in G_1. Applying the same technique to G_R leads to a cubic graph G_2 together a path Q of length 3 (with the edges x_Ry_R and the edges $\{x_Rx_B, y_Ry_B\}$ or $\{x_Ry_R, y_Rx_R\}$. It is a routine matter to check that G is obtained from G_1 and

G_2 via an odd PQ-construction. $\square$

Theorem 7.9 *The following decision problem is NP-complete:*

> **Instance :** *A cubic graph G on $n = 2k$ vertices.*
>
> **Question :** *Does there exists a linear partition $L = (L_B, L_R)$ of G such that $l(L_B) = l(L_R) = k + 1$?*

Proof Let us consider a cubic graph K on p vertices and let xy be an edge of K. Let $\{a, b, c, d\}$ be a set of four vertices disjoint from $V(K)$. Let H be the cubic graph on $p + 4$ vertices such that $V(H) = V(K) \cup \{a, b, c, d\}$ and $E(H) = (E(K) \setminus \{xy\}) \cup \{xa, ab, ac, yd, db, dc, bc\}$. We note that H has a hamiltonian cycle if and only if K has a hamiltonian cycle containing the edge xy.
Let P be a path of length 3 in H on $\{a, b, c, d\}$ (i.e. $P = [a, b, c, d]$ or $P = [a, c, b, d]$). Let us consider a disjoint copy H' of H and Q the corresponding copy of P. Let G be the cubic graph on $n = 2p + 4$ vertices obtained by PQ-construction (see Definition 7.4 and Figure 7).

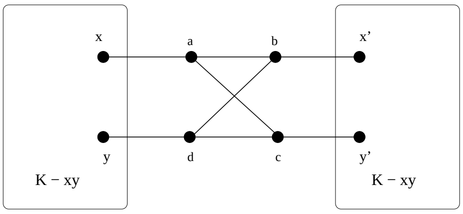

Figure 7: G = PQ(H,H')

Now, let us consider a hamiltonian cycle of K containing xy. Thus, H has a hamiltonian cycle containing bc and H' has a hamiltonian cycle containing $b'c'$. We can provide H (respectively, H') with a linear partition a forest of which is a hamiltonian path, the other forest being the union of a matching and a unique path of length 3, P (respectively, Q). By Proposition 7.7 we obtain a linear partition $L = (L_B, L_R)$ of G such that $l(L_B) = l(L_R) = k + 1$ (with $k = p + 2 = \frac{n}{2}$).
Conversely, suppose that G has a linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = p + 3$. Let P be the unique path of length $p + 3$ of L_B (see Lemma 7.3). Suppose, without loss of generality, that P intersects $V(H) \setminus \{a, b, c, d\}$. If P contains a vertex of $V(H') \setminus \{a, b, c, d\}$ then P contains one of the four paths $[x, a, b, x']$, $[x, a, c, y']$, $[y, d, b, x']$ or $[y, d, c, y']$ (see figure 7). For instance, suppose that P contains $[x, a, b, x']$. Thus, edges ac and bd belongs to L_R and, either $[a, c, d, b]$ is a path of L_R or L_R contains two distinct paths of length at least 2 (containing respectively $[b, d, y]$ and $[a, c, y']$). Since L_R is made of a path Q of length $p + 3$ and a matching of $\frac{p}{2}$ edges, we have a contradiction. Analogously, we see that the paths $[x, a, c, y']$, $[y, d, b, x']$ and $[y, d, c, y']$ are not subpath of P. Hence, $V(P) = V(H) = V(K) \cup \{a, b, c, d\}$. By Proposition 7.8 H is a hamiltonian cubic graph, that is K has a hamiltonian cycle containing xy.

So, we have proved that K has a hamiltonian cycle containing xy if and only if $G = PQ(H, H')$ has a linear partition $L = (L_B, L_R)$ such that $l(L_B) = l(L_R) = p + 3$. It is known that the Hamiltonian Cycle problem remains NP-complete for cubic graphs (see [17]), and it is easy to see that the "Hamiltonian Cycle through a given edge in a cubic graph" decision problem is also NP-complete. The transformation from this last problem to the considered decision problem is clearly a polynomial-time transformation, and the considered decision problem is clearly in NP. So, our decision problem is NP-complete. □

7.2 Knowing that a graph is a Jaeger's graph.

By Shaefer's result [26], we know that it is NP-complete to recognize a Jaeger's graph. Assume that G is a Jaeger's graph, is it difficult to find a perfect matching union of two disjoint strong matchings? We do not have, in general, the answer. However for the particular class of cubic graphs having a 2-factor of squares, this can be done easily (see the proof of Theorem 2.12).

Proposition 7.10 *Let G be a cubic graph with a 2-factor of squares. Then we can find in $O(n)$ time a matching divided into two strong matchings with the same size.*

8 Optimization

As pointed out in proposition 3.5 we can switch from one linear partition $L = (L_B, L_R)$ to another $L' = (L'_1, L'_2)$ with a local transformation as soon as at least one path of L_B (respectively L_R) has two internal vertices which are the end vertices of to two distinct paths of L_R (respectively L_B). Since it is an easy matter (from the complexity point of view, see Theorem 7.1), to construct a linear partition, this local exchange suggests to explore the set of linear partitions in order to maximize some objective function such as $l(L)$ with a simulated annealing approach.
In [8] experimental results have been obtained in order to get a longest path in a "random" cubic hamiltonian graph on at most 500 vertices, showing that this approximation strategy is efficient.

However, we are faced with a difficult problem:

Question 8.1 Is it true that we can reach any linear partition from any other one by using the local exchange described in Proposition 3.5?

Unfortunately, the answer to this question is negative as shown by the following example of Jaeger's graph described in figure 8. In (a), every path of the linear forests has length 3 and the two internal vertices of any such path are the end vertices of another path of length 3 of the linear partition, so we cannot apply a local transformation. Since there is no local exchange possible from (a), the two linear partitions (a) and (b) are in distinct equivalent classes (where an equivalent class is merely the set of all linear partitions which can be obtained from each other by our local exchange).

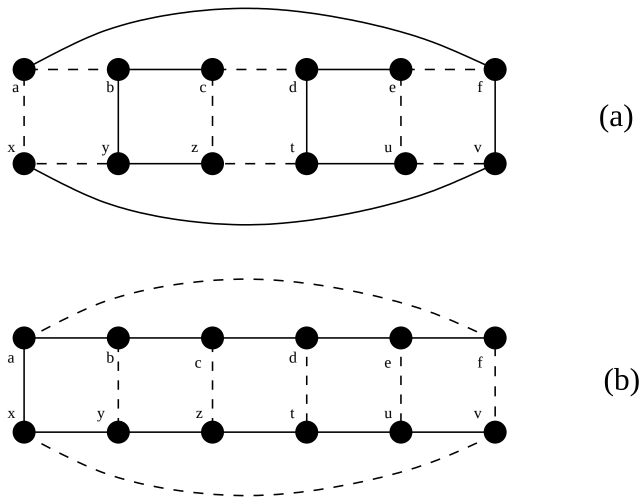

Figure 8: No local exchange

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