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On a family of cubic graphs containing the flower snarks

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On a family of cubic graphs containing the flower snarks

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Abstract

We consider cubic graphs formed with $k \geq 2$ disjoint claws $C_i \sim K_{1,3}$ ($0 \leq i \leq k-1$) such that for every integer i modulo k the three vertices of degree 1 of C_i are joined to the three vertices of degree 1 of C_{i-1} and joined to the three vertices of degree 1 of C_{i+1}. Denote by t_i the vertex of degree 3 of C_i and by T the set $\{t_1, t_2, ..., t_{k-1}\}$. In such a way we construct three distinct graphs, namely $FS(1,k)$, $FS(2,k)$ and $FS(3,k)$. The graph $FS(j,k)$ ($j \in \{1,2,3\}$) is the graph where the set of vertices $\cup_{i=0}^{i=k-1} V(C_i) \setminus T$ induce j cycles (note that the graphs $FS(2, 2p+1)$, $p \geq 2$, are the flower snarks defined by Isaacs [8]). We determine the number of perfect matchings of every $FS(j,k)$. A cubic graph G is said to be 2*-factor hamiltonian* if every 2-factor of G is a hamiltonian cycle. We characterize the graphs $FS(j,k)$ that are 2-factor hamiltonian (note that $FS(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]). A *strong matching* M in a graph G is a matching M such that there is no edge of $E(G)$ connecting any two edges of M. A cubic graph having a perfect matching union of two strong matchings is said to be a *Jaeger's graph*. We characterize the graphs $FS(j,k)$ that are Jaeger's graphs.

Key words: cubic graph; perfect matching; strong matching; counting; hamiltonian cycle; 2-factor hamiltonian

1 Introduction

The complete bipartite graph $K_{1,3}$ is called, as usually, a *claw*. Let k be an integer ≥ 2 and let G be a cubic graph on $4k$ vertices formed with k disjoint claws $C_i = \{x_i, y_i, z_i, t_i\}$ ($0 \leq i \leq k-1$) where t_i (the *center* of C_i) is joined to the three independent vertices x_i, y_i and z_i (the *external* vertices of C_i). For every integer i modulo k C_i has three neighbours in C_{i-1} and three neighbours in C_{i+1}. For any integer $k \geq 2$ we shall denote the set of integers modulo k as $\mathbf{Z}_k$. In the sequel of this paper indices i of claws C_i belong to $\mathbf{Z}_k$.

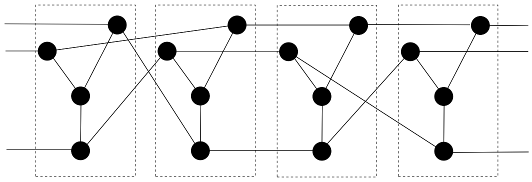

Fig. 1. Four consecutive claws

By renaming some external vertices of claws we can suppose, without loss of generality, that $\{x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}\}$ are edges for any i distinct from $k-1$. That is to say the subgraph induced on $X = \{x_0, x_1, \ldots, x_{k-1}\}$ (respectively $Y = \{y_0, y_1, \ldots, y_{k-1}\}$, $Z = \{z_0, z_1, \ldots, z_{k-1}\}$) is a path or a cycle (as induced subgraph of G). Denote by T the set of the internal vertices $\{t_0, t_1, \ldots, t_{k-1}\}$.

Up to isomorphism, the matching joining the external vertices of C_{k-1} to those of C_0 determines the graph G. In this way we construct essentially three distinct graphs, namely $FS(1,k)$, $FS(2,k)$ and $FS(3,k)$. The graph $FS(j,k)$ ($j \in \{1,2,3\}$) is the graph where the set of vertices $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces j cycles. We note that $FS(3,2)$ and $FS(2,2)$ are multigraphs, and that $FS(1,2)$ is isomorphic to the cube. For $k \geq 3$ and any $j \in \{1,2,3\}$ the graph $FS(j,k)$ is a simple cubic graph. When k is odd, $FS(2,k)$ is the graph known as the flower snark [8].

By using an ad hoc translation of the indices of claws (and of their vertices) and renaming some external vertices of claws, we see that for any reasoning about a sequence of $h \geq 3$ consecutive claws $(C_i, C_{i+1}, C_{i+2}, \ldots, C_{i+h-1})$ there is no loss of generality to suppose that $0 \leq i < i+h-1 \leq k-1$. For a sequence of claws $(C_p, ..., C_r)$ with $0 \leq p < r \leq k-1$, since 0 is a possible value for subscript p and since $k-1$ is a possible value for subscript r, it will be useful from time to time to denote by x'_{p-1} the neighbour in C_{p-1} of the vertex x_p of C_p (recall that $x'_{p-1} \in \{x_{k-1}, y_{k-1}, z_{k-1}\}$ if $p = 0$), and to denote by x'_{r+1} the neighbour in C_{r+1} of the vertex x_r of C_r (recall that $x'_{r+1} \in \{x_0, y_0, z_0\}$ if $r = k-1$). We shall use of analogous notations for neighbours of y_p, z_p, y_r and z_r.

We shall prove in the following lemma that there are essentially two types of perfect matchings in $FS(j,k)$.

Lemma 1 *Let $G \in FS(j,k)$ ($j \in \{1,2,3\}$) and let M be a perfect matching of G. Then the $2-$factor $G \setminus M$ induces a path of length 2 and an isolated vertex in each claw C_i ($i \in \mathbf{Z}_k$) and M verifies one (and only one) of the three following properties :*

i) For every i in $\mathbf{Z}_k$ M contains exactly one edge joining the claw C_i to the claw C_{i+1},
ii) For every even i in $\mathbf{Z}_k$ M contains exactly two edges between C_i and C_{i+1} and none between C_{i-1} and C_i,

iii) For every odd i in $\mathbf{Z}_k$ M contains exactly two edges between C_i and C_{i+1} and none between C_{i-1} and C_i.

Moreover, when k is odd M satisfies only item i).

Proof Let M be a perfect matching of $G \in FS(j,k)$ for some $j \in \{1,2,3\}$. Since M contains exactly one edge of each claw, it is obvious that $G \setminus M$ induces a path of length 2 and an isolated vertex in each claw C_i.

For each claw C_i of G the vertex t_i must be saturated by an edge of M whose end (distinct from t_i) is in $\{x_i, y_i, z_i\}$. Hence there are exactly two edges of M having one end in C_i and the other in $C_{i-1} \cup C_{i+1}$.

If there are two edges of M between C_i and C_{i+1} then there is no edge of M between C_{i-1} and C_i. If there are are two edges of M between C_{i-1} and C_i then there is no edge of M between C_i and C_{i+1}. Hence, we get *ii)* or *iii)* and we must have an even number k of claws in G.

Assume now that there is only one edge of M between C_{i-1} and C_i. Then there exists exactly one edge between C_i and C_{i+1} and, extending this trick to each claw of G, we get *i)* when k is even or odd. □

Definition 2 We say that a perfect matching M of $FS(j,k)$ is of *type* 1 in Case i) of Lemma 1 and of *type* 2 in Cases ii) and iii). If neccessary, to distinguish Case ii) from Case iii) we shall say *type* 2.0 in Case ii) and *type* 2.1 in Case iii). We note that the numbers of perfect matchings of type 2.0 and of type 2.1 are equal.

Notation : The length of a path P (respectively a cycle Γ) is denoted by $l(P)$ (respectively $l(\Gamma)$).

2 Counting perfect matchings of $FS(j,k)$

We shall say that a vertex v of a cubic graph G is *inflated* into a triangle when we construct a new cubic graph G' by deleting v and adding three new vertices inducing a triangle and joining each vertex of the neighbourood $N(v)$ of v to a single vertex of this new triangle. We say also that G' is obtain from G by a *triangular extension.* The converse operation is the *contraction* or *reduction* of the triangle. The number of perfect matchings of G is denoted by $\mu(G)$.

Lemma 3 *Let G be a bipartite cubic graph and let $\{V_1, V_2\}$ be the bipartition of its vertex set. Assume that each vertex in $W_1 \subseteq V_1$ is inflated into a triangle* *and let G' be the graph so obtained. Then $\mu(G) = \mu(G')$.*

Proof Note that $\{V_1, V_2\}$ is a balanced bipartition and, by König's Theorem, the graph G is a cubic 3-edge colourable graph. So, G' is also a cubic 3-edge colourable graph (hence, G and G' have perfect matchings). Let M be a perfect matching of G'. Each vertex of $V_1 \setminus W_1$ is saturated by an edge whose second end vertex is in V_2. Let $A \subseteq V_2$ be the set of vertices so saturated in V_2. Assume that some triangle of G' is such that the three vertices are saturated by three edges having one end in the triangle and the second one in V_2. Then we need to have at least $|W_1| + 2$ vertices in $V_2 \setminus A$, a contradiction. Hence, M must have exactly one edge in each triangle and the contraction of each triangle in order to get back G transforms M in a perfect matching of G. Conversely, each perfect matching of G leads to a unique perfect matching of G' and we obtain the result. $\square$

Let us denote by $\mu(j,k)$ the number of perfect matchings of $FS(j,k)$, $\mu_1(j,k)$ its number of perfect matchings of type 1 and $\mu_2(j,k)$ its number of perfect matchings of type 2.

Lemma 4 *We have*

- $\mu(1,3) = \mu_1(1,3) = 9$
- $\mu(2,3) = \mu_1(2,3) = 8$
- $\mu(3,3) = \mu_1(3,3) = 6$
- $\mu(1,2) = 9,\ \mu_1(1,2) = 3$
- $\mu(2,2) = 10,\ \mu_1(2,2) = 4$
- $\mu(3,2) = 12,\ \mu_1(3,2) = 6$

Proof The cycle containing the external vertices of the claws of the graph $FS(1,3)$ is $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, x_0$. Consider a perfect matching M containing the edge t_0x_0. There are two cases: *i)* $x_1x_2 \in M$ and *ii)* $x_1t_1 \in M$. In Case *i)* we must have $y_0y_1, t_1z_1, t_2z_2, z_0y_2 \in M$. In Case *ii)* there are two sub-cases: *ii).a* $x_2y_0 \in M$ and *ii).b* $x_2t_2 \in M$. In Case *ii).a* we must have $y_1y_2, t_2z_2, z_0z_1 \in M$ and in Case *ii).b* we must have $y_0y_1, y_2z_0, z_1z_2 \in M$. Thus, there are exactly 3 distinct perfect matching containing t_0x_0. By symmetry, for $i = 1, 2$ there are 3 distinct perfect matchings containing t_ix_i, therefore $\mu(1,3) = 9$.

It is well known that the Petersen graph has exactly 6 perfect matchings. Since $FS(2,3)$ is obtained from the Petersen graph by inflating a vertex in a triangle these 6 perfect matchings lead to 6 perfect matchings of $FS(2,3)$. We have two new perfect matchings when considering the three edges connected to this triangle (we have two ways to include these edges in a perfect matching). Hence $\mu(2,3) = 8$.

$FS(3,3)$ is obtained from $K_{3,3}$ by inflating three vertices in the same colour of the bipartition. Since $K_{3,3}$ has 6 perfect matchings, applying Lemma 3 we get immediately the result for $\mu(3,3)$.

Is is a routine matter to obtain the values for $FS(j,2)$ $(j \in \{1,2,3\})$. $\square$

Theorem 5 *The numbers $\mu(i,k)$ of perfect matchings of $FS(i,k)$ $(i \in \{1,2,3\})$ are given by:*

When $k \equiv 1(2)$

- $\mu(2,k) = 2^k$
- $\mu(1,k) = 2^k + 1$
- $\mu(3,k) = 2^k - 2$

When $k \equiv 0(2)$

- $\mu(2,k) = 2 \times 3^{\frac{k}{2}} + 2^k$
- $\mu(1,k) = 2 \times 3^{\frac{k}{2}} + 2^k - 1$
- $\mu(3,k) = 2 \times 3^{\frac{k}{2}} + 2^k + 2$

Proof We shall prove this result by induction on k and we distinguish the case "k odd" and the case "k even".

The following trick will be helpful. Let $i \neq 0$ and let C_{i-2}, C_{i-1}, C_i and C_{i+1} be four consecutive claws of $FS(j,k)$ $(j \in \{1,2,3\})$. We can delete C_{i-1} and C_i and join the three external vertices of C_{i-2} to the three external vertices of C_{i+1} by a matching in such a way that the resulting graph is $FS(j', k-2)$. We have three distinct ways to reduce $FS(j,k)$ into $FS(j', k-2)$ when deleting C_{i-1} and C_i.

Case 1: We add the edges $\{x_{i-2}x_{i+1}, y_{i-2}y_{i+1}, z_{i-2}z_{i+1}\}$ and we get $G_1 = FS(j_1, k-2)$

Case 2: We add the edges $\{x_{i-2}y_{i+1}, y_{i-2}z_{i+1}, z_{i-2}x_{i+1}\}$ and we get $G_2 = FS(j_2, k-2)$.

Case 3: We add the edges $\{x_{i-2}z_{i+1}, y_{i-2}x_{i+1}, z_{i-2}y_{i+1}\}$ and we get $G_3 = FS(j_3, k-2)$.

Following the cases, we shall precise the values of j_1, j_2 and j_3.

It is an easy task to see that each perfect matching of type 1 of $FS(j,k)$ leads to a perfect matching of G_1, G_2 or G_3 and, conversely, each perfect matching of type 1 of G_1 allows us to construct 2 distinct perfect matchings of type 1 of $FS(j,k)$, while each perfect matching of type 1 of G_2 and G_3 allows us to construct 1 perfect matching of type 1 of $FS(j,k)$.

We have

$$\mu_1(2,k) = 2\mu_1(G_1) + \mu_1(G_2) + \mu_1(G_3) \tag{1}$$

CLAIM 1 $\mu_1(2,k) = 2^k$

Proof Since the result holds for $FS(2,3)$ and $FS(2,2)$ by Lemma 4, in order to prove the result by induction on the number k of claws, we assume that the property holds for $FS(2,k-2)$ with $k-2 \geq 2$.

In that case G_1, G_2 and G_3 are isomorphic to $FS(2,k-2)$. Using Equation 1 we have, as claimed

$$\mu_1(2,k) = 4\mu_1(2,k-2) = 2^k$$

□

CLAIM 2 $\mu_1(1,k) = 2^k + 1$ *and* $\mu_1(3,k) = 2^k - 2$

Proof Since the result holds for $FS(1,3)$ $FS(1,2)$, $FS(3,3)$ $FS(3,2)$by Lemma 4, in order to prove the result by induction on the number k of claws, we assume that the property holds for $FS(1,k-2)$, and $FS(3,k-2)$ with $k-2 \geq 2$.

When considering $FS(1,k)$, G_1 is isomorphic to $FS(1,k-2)$, and among G_2 and G_3 one of them is isomorphic to $FS(3,k-2)$ and the other to $FS(1,k-2)$. In the same way, when considering $FS(3,k)$, G_1 is isomorphic to $FS(3,k-2)$, and G_2 and G_3 are isomorphic to $FS(1,k-2)$.

Using Equation 1 we have,

$$\mu_1(1,k) = 2\mu_1(1,k-2) + \mu_1(1,k-2) + \mu_1(3,k-2)$$

and

$$\mu_1(1,k) = 2(2^{k-2}+1) + 2^{k-2} + 1 + 2^{k-2} - 2 = 2^k + 1$$
$$\mu_1(3,k) = 2(2^{k-2}-2) + 2^{k-2} + 1 + 2^{k-2} + 1 = 2^k - 2$$

□

When k is odd, we have $\mu_2(j,k) = 0$ by Lemma 1 and hence $\mu(j,k) = \mu_1(j,k)$

When k is even it remains to count the number of perfect matchings of type 2. From Lemma 1, for every tow consecutive claws C_i and C_{i+1}, we have either two edges of M joining the external vertices of C_i to those of C_{i+1} or none. We have 3 ways to choose 2 edges between C_i and C_{i+1}, each choice of these two edges can be completed in a unique way in a perfect matching of the subgraph $C_i \cup C_{i+1}$. Hence we get easily that the number of perfect matchings of type 2 in $FS(j,k)$ ($j \in \{1,2,3\}$) is

$$\mu_2(j,k) = 2 \times 3^{\frac{k}{2}} \tag{2}$$

Using Claims 1 and 2 and Equation 2 we get the results for $\mu(j,k)$ when k is even.

□

3 Some structural results about perfect matchings of FS(j, k)

3.1 Perfect matchings of type 1

Lemma 6 *Let M be a perfect matching of type* 1 *of* $G = FS(j,k)$. *Then the* $2-$*factor* $G \setminus M$ *has exactly one or two cycles and each cycle of* $G \setminus M$ *has at least one vertex in each claw* C_i *(*$i \in \mathbf{Z}_k$*).*

Proof Let M be a perfect matching of type 1 in G. Let us consider the claw C_i for some i in $\mathbf{Z}_k$. Assume without loss of generality that the edge of M contained in C_i is t_ix_i. The cycle of $G \setminus M$ visiting x_i comes from C_{i-1}, crosses C_i by using the vertex x_i and goes to C_{i+1}. By Lemma 1, the path $y_it_iz_i$ is contained in a cycle of $G \setminus M$. The two edges incident to y_i and z_i joining C_i to C_{i-1} (as well as those joining C_i to C_{i+1}) are not contained both in M (since M has type 1). Thus, the cycle of $G \setminus M$ containing $y_it_iz_i$ comes from C_{i-1}, crosses C_i and goes to C_{i+1}. Thus, we have at most two cycles in $G \setminus M$, as claimed, and we can note that each claw must be visited by these cycles. □

Definition 7 Let us suppose that M is a perfect matching of type 1 in $G = FS(j,k)$ such that the $2-$factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. A claw C_i intersected by three vertices of Γ_1 (respectively Γ_2) is said to be Γ_1-*major* (respectively Γ_2-*major).*

Lemma 8 *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the $2-$factor $G \setminus M$ has exactly two cycles. Then, the lengths of these two cycles have the same parity as k, and those lengths are distinct when k is odd.*

Proof Let Γ_1 and Γ_2 be the two cycles of $G \setminus M$. By Lemma 6, for each i in $\mathbf{Z}_k$ these two cycles must cross the claw C_i. Let k_1 be the number of Γ_1-major claws and let k_2 be the number of Γ_2-major claws. We have $k_1 + k_2 = k$, $l(\Gamma_1) = 3k_1 + k_2$ and $l(\Gamma_2) = 3k_2 + k_1$. When k is odd, we must have either k_1 odd and k_2 even, or k_1 even and k_2 odd. Then Γ_1 and Γ_2 have distinct odd lengths. When k is even, we must have either k_1 and k_2 even, or k_2 and k_1 odd. Then Γ_1 and Γ_2 have even lengths. $\square$

Lemma 9 *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the $2-$factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are two consecutive Γ_1-major claws C_j and C_{j+1} with $j \in \mathbf{Z}_k \setminus \{k-1\}$. Then there is a perfect matching M' of type 1 such that the $2-$factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

a) for $i \in \mathbf{Z}_k \setminus \{j, j+1\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,
b) C_j and C_{j+1} are Γ'_2-major,
c) $l(\Gamma'_1) = l(\Gamma_1) - 4$ and $l(\Gamma'_2) = l(\Gamma_2) + 4$.

Proof Consider the claws C_j and C_{j+1}. Since C_j is a Γ_1-major claw suppose without loss of generality that t_jz_j belongs to M and that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ where x'_{j-1} denotes the neighbour of x_j in C_{j-1} (then x_jx_{j+1} belongs to M). Since C_{j+1} is Γ_1-major and Γ_2 goes through C_j and C_{j+1}, the cycle Γ_1 must contain the path $y_{j+1}t_{j+1}x_{j+1}x'_{j+2}$ where x'_{j+2} denotes the neighbour of x_{j+1} in C_{j+2} (then M contains $t_{j+1}z_{j+1}$ and $y_{j+1}y'_{j+2}$). Denote by P_1 the path $x'_{j-1}x_jt_jy_jy_{j+1}t_{j+1}x_{j+1}x'_{j+2}$. Note that Γ_2 contains the path $P_2 = z'_{j-1}z_jz_{j+1}z'_{j+1}$ where z'_{j-1} and z'_{j+1} are defined similarly. See to the left part of Figure 2.

Let us perform the following local transformation: delete x_jx_{j+1}, t_jz_j and $t_{j+1}z_{j+1}$ from M and add z_jz_{j+1}, t_jx_j and $t_{j+1}x_{j+1}$. Let M' be the resulting perfect matching. Then the subpath P_1 of Γ_1 is replaced by $P'_1 = x'_{j-1}x_jx_{j+1}x'_{j+2}$ and the subpath P_2 of Γ_2 is replaced by $P'_2 = z'_{j-1}z_jt_jy_jy_{j+1}t_{j+1}z_{j+1}z'_{j+2}$ (see Figure 2). We obtain a new 2-factor containing two new cycles Γ'_1 and Γ'_2. Note that C_j and C_{j+1} are Γ'_2-major claws and for i in $\mathbf{Z}_k \setminus \{j, j+1\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). The length of Γ_1 (now Γ'_1) decreases of 4 units while the length of Γ_2 (now Γ'_2) increases of 4 units. $\square$

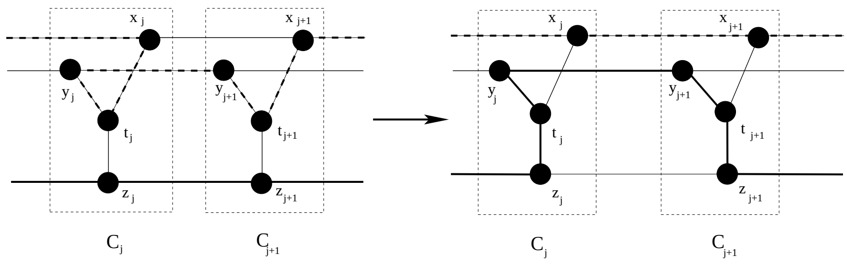

Fig. 2. Local transformation of type 1

The operation depicted in Lemma 9 above will be called a *local transformation of type 1*.

Lemma 10 *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the $2-$factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are three consecutive claws C_j, C_{j+1} and C_{j+2} with j in $\mathbf{Z}_k \setminus \{k-1, k-2\}$ such that C_j and C_{j+2} are Γ_1-major and C_{j+1} is Γ_2-major. Then there is a perfect matching M' of type 1 such that the $2-$factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

a) for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,
b) C_j and C_{j+2} are Γ'_2-major and C_{j+1} is Γ'_1-major,
c) $l(\Gamma'_1) = l(\Gamma_1) - 2$ and $l(\Gamma'_2) = l(\Gamma_2) + 2$.

Proof Since C_j is Γ_1-major, as in the proof of Lemma 9 suppose that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ (that is edges t_jz_j and x_jx_{j+1} belong to M). Since C_{j+1} is Γ_2-major the cycle Γ_1 contains the edge $y_{j+1}y_{j+2}$. Then we see that Γ_1 contains the path $Q_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}t_{j+2}z_{j+2}z'_{j+3}$ and that Γ_2 contains the path $Q_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}x'_{j+3}$. Note that $y_{j+1}t_{j+1}$, $z_{j+1}z_{j+2}$ and $t_{j+2}x_{j+2}$ belong to M.

Let us perform the following local transformation: delete t_jz_j, x_jx_{j+1}, $z_{j+1}z_{j+2}$ and $x_{j+2}t_{j+2}$ from M and add x_jt_j, z_jz_{j+1}, $x_{j+1}x_{j+2}$ and $z_{j+2}t_{j+2}$ to M. Let M' be the resulting perfect matching. Then the subpath Q_1 of Γ_1 is replaced by $Q'_1 = x'_{j-1}x_jx_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$ and the subpath Q_2 of Γ_2 is replaced by $Q'_2 = z'_{j-1}z_jt_jy_jy_{j+1}y_{j+2}t_{j+2}x_{j+2}x'_{j+3}$ (see Figure 3). We obtain a new 2-factor containing two new cycles named Γ'_1 and Γ'_2. Note that C_j and C_{j+2} are now Γ'_2-major claws and C_{j+1} is Γ'_1-major. The length of Γ_1 decreases of 2 units while the length of Γ_2 increases of 2 units. It is clear that for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). $\square$

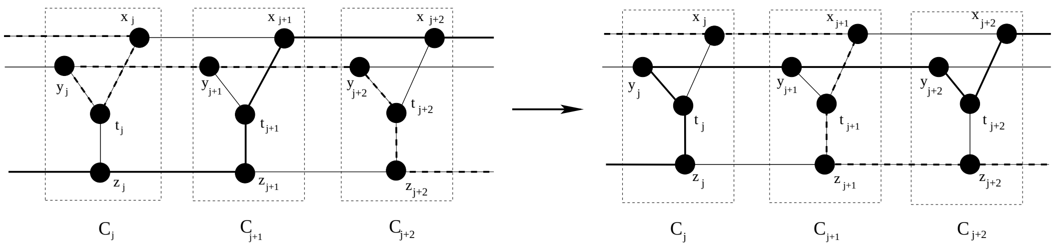

Fig. 3. Local transformation of type 2

The operation depicted in Lemma 10 above will be called a *local transformation of type 2.*

Lemma 11 *Let M be a perfect matching of type* 1 *of $G = FS(j,k)$ such that the $2-$factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are three consecutive claws C_j, C_{j+1} and C_{j+2} with j in $\mathbf{Z}_k \setminus \{k-1, k-2\}$ such that C_{j+1} and C_{j+2} are Γ_2-major and C_j is Γ_1-major. Then there is a perfect matching M' of type* 1 *such that the $2-$factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

a) for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,
b) C_j and C_{j+1} are Γ'_2-major and C_{j+2} is Γ'_1-major,
c) $l(\Gamma'_1) = l(\Gamma_1)$ and $l(\Gamma'_2) = l(\Gamma_2)$.

Proof Since C_j is Γ_1-major, as in the proof of Lemma 9 suppose that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ (that is edges t_jz_j and x_jx_{j+1} belong to M). Since C_{j+1} and C_{j+2} are Γ_2-major, the unique vertex of C_{j+1} (respectively C_{j+2}) contained in Γ_1 is y_{j+1} (respectively y_{j+2}). Note that the perfect matching M contains the edges t_jz_j, x_jx_{j+1}, $t_{j+1}y_{j+1}$, $z_{j+1}z_{j+2}$ and $t_{j+2}y_{j+2}$. Then the path $R_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}y'_{j+3}$ is a subpath of Γ_1 and the path $R_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}t_{j+2}z_{j+2}z'_{j+3}$ is a subpath of Γ_2. See to the left part of Figure 4.

Let us perform the following local transformation: delete t_jz_j, x_jx_{j+1}, $t_{j+1}y_{j+1}$, $z_{j+1}z_{j+2}$ and $t_{j+2}y_{j+2}$ from M and add x_jt_j, z_jz_{j+1}, $t_{j+1}x_{j+1}$, $y_{j+1}y_{j+2}$ and $t_{j+2}z_{j+2}$. Let M' be the resulting perfect matching. Then the subpath R_1 of Γ_1 is replaced by $R'_1 = x'_{j-1}x_jx_{j+1}x_{j+2}t_{j+2}y_{j+2}y'_{j+3}$ and the subpath R_2 of Γ_2 is replaced by $R'_2 = z'_{j-1}z_jt_jy_jy_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$. We obtain a new 2-factor containing two new cycles named Γ'_1 and Γ'_2 such that $l(\Gamma'_1) = l(\Gamma_1)$ and $l(\Gamma'_2) = l(\Gamma_2)$ (see Figure 4). It is clear that for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). Note that C_j and C_{j+1} are Γ'_2-major and C_{j+2} is Γ'_1-major. $\square$

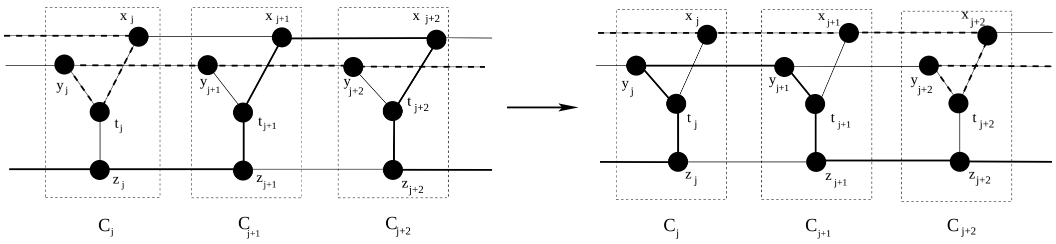

Fig. 4. Local transformation of type 3

The operation depicted in Lemma 11 above will be called a *local transformation of type 3.*

Lemma 12 *Let M be a perfect matching of type* 1 *of* $G = FS(j,k)$ *such that the* $2-$*factor* $G \setminus M$ *has exactly two cycles* Γ_1 *and* Γ_2 *such that* $l(\Gamma_1) \leq l(\Gamma_2)$ *and* $l(\Gamma_2)$ *is as great as possible. Then there exists at most one* Γ_1*-major claw.*

Proof Suppose, for the sake of contradiction, that there exist at least two Γ_1-major claws. Since $l(\Gamma_2)$ is maximum, by Lemma 9 these claws are not consecutive. Then consider two Γ_1-major claws C_i and C_{i+h+1} (with $h \geq 1$) such that the h consecutive claws $(C_{i+1}, \dots, C_{i+h})$ are Γ_2-major. Since $l(\Gamma_2)$ is maximum, by Lemma 10 the number h is at least 2. Then by applying $r = \lfloor \frac{h}{2} \rfloor$ consecutive local transformations of type 3 (Lemma 11) we obtain a perfect matching $M^{(r)}$ such that the $2-$factor $G \setminus M^{(r)}$ has exactly two cycles $\Gamma_1^{(r)}$ and $\Gamma_2^{(r)}$ with $l(\Gamma_1^{(r)}) = l(\Gamma_1)$ and $l(\Gamma_2^{(r)}) = l(\Gamma_2)$ and such that $C_{i+2\lfloor \frac{h}{2} \rfloor}$ and C_{i+h+1} are $\Gamma_1^{(r)}$-major. Since $l(\Gamma_2^{(r)})$ is maximum, we can conclude by Lemma 9 and by Lemma 10 that h is neither even nor odd, a contradiction. $\square$

3.2 Perfect matchings of type 2

We give here a structural result about perfect matchings of type 2 in $G = FS(j,k)$.

Lemma 13 *Let M be a perfect matching of type* 2 *of* $G = FS(j,k)$ *(with* $k \geq 4$*). Then the* $2-$*factor* $G \setminus M$ *has exactly one cycle of even length* $l \geq k$ *and a set of* p *cycles of length* 6 *where* $l + 6p = 4k$ *(with* $0 \leq p \leq \frac{k}{2}$*).*

Proof Let M be a perfect matching of type 2 in G. By Lemma 1 the number k of claws is even. Let i in $\mathbf{Z}_k$ such that there are two edges of M between C_{i-1} and C_i. There are no edges of M between C_i and C_{i+1} and two edges of M between C_{i+1} and C_{i+2}. We may consider that $0 \leq i < k-1$.
For $j \in \{i, i+2, i+4, \dots\}$ we denote by e_j the unique edge of $G \setminus M$ having

one end vertex in C_{j-1} and the other in C_j. Let us denote by A the set $\{e_i, e_{i+2}, e_{i+4}, \ldots\}$. We note that $\mid A \mid = \frac{k}{2}$.
Assume without loss of generality that the two edges of M between C_{i-1} and C_i have end vertices in C_i which are x_i and y_i (then z_i is the end vertex of e_i in C_i). Two cases may now occur.

Case 1: *The end vertices in C_{i+1} of the two edges of M between C_{i+1} and C_{i+2} are x_{i+1} and y_{i+1}* (then z_{i+1} is the end vertex of e_{i+2} in C_{i+1}).
In that case the $2-$ factor $G \backslash M$ contains the cycle of length 6 $x_i x_{i+1} t_{i+1} y_{i+1} y_i t_i$ while the edge $z_i z_{i+1}$ of $G \setminus M$ relies e_i and e_{i+2}.

Case 2: *The end vertices in C_{i+1} of the two edges of M between C_{i+1} and C_{i+2} are y_{i+1} and z_{i+1} (respectively x_{i+1} and z_{i+1}).* Then x_{i+1} (respectively y_{i+1}) is the end vertex of e_{i+2} in C_{i+1}.
In that case the edges e_i and e_{i+2} are connected in $G \setminus M$ by the path $z_i z_{i+1} t_{i+1} y_{i+1} y_i t_i x_i x_{i+1}$ (respectively $z_i z_{i+1} t_{i+1} x_{i+1} x_i t_i y_i y_{i+1}$).

The same reasoning can be done for $\{e_{i+2}, e_{i+4}\}$, $\{e_{i+4}, e_{i+6}\}$, and so on. Then, we see that the set A is contained in a unique cycle Γ of $G \setminus M$ which crosses each claw. Thus, the length l of Γ is at least k. More precisely, each e_j in A contributes for 1 in l, in Case 1 the edge $z_i z_{i+1}$ contributes for 1 in l and in Case 2 the path $z_i z_{i+1} t_{i+1} y_{i+1} y_i t_i x_i x_{i+1}$ contributes for 7 in l. Let us suppose that Case 1 appears p times $(0 \leq p \leq \frac{k}{2})$, that is to say $G \setminus M$ contains p cycles of length 6. Since Case 2 appears $\frac{k}{2} - p$ times, the length of Γ is $l = \frac{k}{2} + p + 7(\frac{k}{2} - p) = 4k - 6p$. □

Remark 14 If k is even then by Lemmas 6, 8 and 13 $FS(j,k)$ has an even 2-factor. That is to say $FS(j,k)$ is a cubic 3-edge colourable graph .

4 Perfect matchings and hamiltonian cycles of $F(j,k)$

4.1 Perfect matchings of type 1 and hamiltonicity

Theorem 15 *Let M be a perfect matching of type* 1 *of* $G = FS(j,k)$. *Then the* 2*-factor* $G \setminus M$ *is a hamiltonian cycle except for* k *odd and* $j = 2$, *and for* k *even and* $j = 1$ or 3.

Proof Suppose that there exists a perfect matching M of type 1 of G such that $G \setminus M$ is not a hamiltonian cycle. By Lemma 6 and Lemma 8 the 2-factor $G \setminus M$ is made of exactly two cycles Γ_1 and Γ_2 whose lengths have the same parity as k. Without loss of generality we suppose that $l(\Gamma_1) \leq l(\Gamma_2)$.

Assume moreover that among the perfect matchings of type 1 of G such that the 2−factor $G \setminus M$ is composed of two cycles, M has been chosen in such a way that the length of the longest cycle Γ_2 is as great as possible. By Lemma 12 there exists at most one Γ_1-major claw.

Case 1: There exists one Γ_1-major claw.

Without loss of generality, suppose that C_0 is intersected by Γ_1 in $\{y_0, t_0, x_0\}$ and that $y'_{k-1}y_0$ belongs to Γ_1. Since for every $i \neq 0$ the claw C_i is Γ_2-major, Γ_1 contains the vertices $y_0, t_0, x_0, x_1, x_2, \ldots, x_{k-1}$.

• If $k = 2r + 1$ with $r \geq 1$ then Γ_2 contains the path

$$z_0z_1t_1y_1y_2t_2z_2...z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}.$$

Thus, y_0x_{k-1}, x_0y_{k-1}, z_0z_{k-1} are edges of G. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces two cycles, that is to say $j = 2$ and $G = FS(2, k)$.

• If $k = 2r + 2$ with $r \geq 1$ then Γ_2 contains the path

$$z_0z_1t_1y_1y_2t_2z_2...z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}z_{2r+1}t_{2r+1}y_{2r+1}.$$

Thus, x_0z_{k-1}, y_0x_{k-1} and z_0y_{k-1} are edges. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces one cycle, that is to say $j = 1$ and $G = FS(1, k)$.

Case 2: There is no Γ_1-major claw.

Suppose that x_0 belongs to Γ_1. Then, Γ_1 contains $x_0, x_1, ..., x_{k-1}$.

• If $k = 2r + 1$ with $r \geq 1$ then Γ_2 contains the path

$$y_0t_0z_0z_1t_1y_1y_2...z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}.$$

Thus, x_0x_{k-1}, y_0z_{k-1} and z_0y_{k-1} are edges of G and the set $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces two cycles,that is to say $j = 2$ and $G = FS(2, k)$.

• If $k = 2r + 2$ with $r \geq 1$ then Γ_2 contains the path

$$y_0t_0z_0z_1t_1y_1y_2...y_{2r}t_{2r}z_{2r}z_{2r+1}t_{2r+1}y_{2r+1}.$$

Thus, x_0x_{k-1}, y_0y_{k-1} and z_0z_{k-1} are edges. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces three cycles, that is to say $j = 3$ and $G = FS(3, k)$.

□

Definition 16 A cubic graph G is said to be 2-*factor hamiltonian* [6] if every 2-factor of G is a hamiltonian cycle (or equivalently, if for every perfect matching M of G the 2-factor $G \setminus M$ is a hamiltonian cycle).

By Theorem 15 for any odd $k \geq 3$ and $j \in \{1,3\}$ or for any even k and $j = 2$, and for every perfect matching M of type 1 in $FS(j,k)$ the 2-factor $FS(j,k) \setminus M$ is a hamiltonian cycle. By Lemma 13 $FS(2,k)$ ($k \geq 4$) may have a perfect matching M of type 2 such that the 2-factor $FS(2,k) \setminus M$ is not a hamiltonian cycle (it may contains cycles of length 6).

Then we have the following.

Corollary 17 *A graph $G = FS(j,k)$ is 2-factor hamiltonian if and only if k is odd and $j = 1$ or 3.*

We note that $FS(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]. We shall examine others known results about 2-factor hamiltonian cubic graphs in Section 5.

Corollary 18 *The chromatic index of a graph $G = FS(j,k)$ is 4 if and only if $j = 2$ and k is odd.*

Proof When $j = 2$ and k is odd, any 2-factor must have at least two cycles, by Theorem 15. Then Lemma 8 implies that any 2-factor is composed of two odd cycles. Hence G has chromatic index 4.

When $j = 1$ or 3 and k is odd by Theorem 15 $FS(j,k)$ is hamiltonian. If k is even then by Lemmas 6, 8 and 13 $FS(j,k)$ has an even 2-factor. $\square$

4.2 Perfect matchings of type 2 and hamiltonicity

At this point of the discourse one may ask what happens for perfect matchings of type 2 in $FS(j,k)$ (k even). Can we characterize and count perfect matchings of type 2, complementary 2-factor of which is a hamiltonian cycle ? An affirmative answer shall be given.

Let us consider a perfect matching M of type 2 in $FS(j,2p)$ with $p \geq 2$. Suppose that there are no edges of M between C_{2i-1} and C_{2i} (for any $i \geq 1$), that is M is a matching of type 2.0 (see Definition 2). Consider two consecutive claws C_{2i} and C_{2i+1} ($0 \leq i \leq p-1$). There are three cases:

Case (x): $\{y_{2i}y_{2i+1}, z_{2i}z_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{x_{2i}t_{2i}, x_{2i+1}t_{2i+1}\}$).

Case (y): $\{x_{2i}x_{2i+1}, z_{2i}z_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{y_{2i}t_{2i}, y_{2i+1}t_{2i+1}\}$).

Case (z): $\{x_{2i}x_{2i+1}, y_{2i}y_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{z_{2i}t_{2i}, z_{2i+1}t_{2i+1}\}$).

The subgraph induced on $C_{2i} \cup C_{2i+1}$ is called a *block*. In Case (x) (respectively Case (y), Case (z)) a block is called a *block of type* X (respectively *block of type* Y, *block of type* Z). Then $FS(j, 2p)$ with a perfect matchings M of type 2.0 can be seen as a sequence of p blocks properly relied. In other words, a perfect matchings M of type 2 in $FS(j, 2p)$ is entirely described by a word of length p on the alphabet of three letters $\{X, Y, Z\}$. The block $C_0 \cup C_1$ is called *initial block* and the block $C_{2p-1} \cup C_{2p}$ is called *terminal block*. These extremal blocks are not considered here as consecutive blocks.

By Lemma 13, $FS(j, 2p) \setminus M$ has no 6-cycles if and only if $FS(j, 2p) \setminus M$ is a unique even cycle. It is an easy matter to prove that two consecutive blocks do not induce a 6-cycle if and only if they are not of the same type. Then the possible configurations for two consecutive blocks are XY, XZ, YX, YZ, ZX and ZY. To eliminate a possible 6-cycle in $C_0 \cup C_{2p-1}$ we have to determine for every $j \in \{1, 2, 3\}$ the forbidden extremal configurations. An extremal configuration shall be denoted by a word on two letters in $\{X, Y, Z\}$ such that the left letter denotes the type of the initial block $C_0 \cup C_1$ and the right letter denotes the type of the terminal block $C_{2p-1} \cup C_{2p}$. We suppose that the extremal blocks are connected for $j = 1$ by the edges $x_{2p-1}z_0$, $y_{2p-1}x_0$ and $z_{2p-1}y_0$, for $j = 2$ by the edges $x_{2p-1}x_0$, $y_{2p-1}z_0$ and $z_{2p-1}y_0$ and for $j = 3$ by the edges $x_{2p-1}x_0$, $y_{2p-1}y_0$ and $z_{2p-1}z_0$. Then, it is easy to verify that we have the following result.

Lemma 19 *Let M be a perfect matching of type* 2.0 *of $G = FS(j, 2p)$ (with $p \geq 2$) such that the* 2*−factor $G \setminus M$ is a hamiltonian cycle. Then the forbidden extremal configurations are*

XY, YZ and ZX for $FS(1, 2p)$,

XX, YZ and ZY for $FS(2, 2p)$,

and XX, YY and ZZ for $FS(3, 2p)$.

Thus, any perfect matching M of type 2.0 of $FS(j, 2p)$ such that the 2−factor $G \setminus M$ is a hamiltonian cycle is totally characterized by a word of length p on the alphabet $\{X, Y, Z\}$ having no two identical consecutive letters and such that the sub-word [initial letter][terminal letter] is not a forbidden configuration. Then, we are in position to obtain the number of such perfect matchings in $FS(j, 2p)$. Let us denote by $\mu'_{2.0}(j, 2p)$ (respectively $\mu'_{2.1}(j, 2p)$, $\mu'_2(j, 2p)$) the number of perfect matchings of type 2.0 (respectively type 2.1, type 2) complementary to a hamiltonian cycle in $FS(j, 2p)$. Clearly $\mu'_2(j, 2p) = \mu'_{2.0}(j, 2p) + \mu'_{2.1}(j, 2p)$ and $\mu'_{2.0}(j, 2p) = \mu'_{2.1}(j, 2p)$.

Theorem 20 *The numbers $\mu_2'(j,2p)$ of perfect matchings of type 2 complementary to hamiltonian cycles in $FS(j,2p)$ ($j \in \{1,2,3\}$) are given by:*

$$\mu_2'(1,2p) = 2^{p+1} + (-1)^{p+1}2,$$

$$\mu_2'(2,2p) = 2^{p+1} \ ,$$

and $$\mu_2'(3,2p) = 2^{p+1} + (-1)^p 4.$$

Proof Consider, as previously, perfect matchings of type 2.0. Let α and β be two letters in $\{X,Y,Z\}$ (not necessarily distinct). Let $A^p_{\alpha\beta}$ be the set of words of length p on $\{X,Y,Z\}$ having no two consecutive identical letters, beginning by α and ending by a letter distinct from β. Denote the number of words in $A^p_{\alpha\beta}$ by $a^p_{\alpha\beta}$. Let $B^p_{\alpha\beta}$ be the set of words of length p on $\{X,Y,Z\}$ having no two consecutive identical letters, beginning by α and ending by β. Denote by $b^p_{\alpha\beta}$ the number of words in $B^p_{\alpha\beta}$.

Clearly, the number of words of length p having no two consecutive identical letters and beginning by α is 2^{p-1}. Then $a^p_{\alpha\beta} + b^p_{\alpha\beta} = 2^{p-1}$. The deletion of the last β of a word in $B^p_{\alpha\beta}$ gives a word in $A^{p-1}_{\alpha\beta}$ and the addition of β to the right of a word in $A^{p-1}_{\alpha\beta}$ gives a word in $B^p_{\alpha\beta}$.

Thus $b^p_{\alpha\beta} = a^{p-1}_{\alpha\beta}$ and for every $p \geq 3$ $a^p_{\alpha\beta} = 2^{p-1} - a^{p-1}_{\alpha\beta}$. We note that $a^2_{\alpha\beta} = 2$ if $\alpha = \beta$, and $a^2_{\alpha\beta} = 1$ if $\alpha \neq \beta$. If $\alpha = \beta$ we have to solve the recurrent sequence : $u_2 = 2$ and $u_p = 2^{p-1} - u_{p-1}$ for $p \geq 3$. If $\alpha \neq \beta$ we have to solve the recurrent sequence : $v_2 = 1$ and $v_p = 2^{p-1} - v_{p-1}$ for $p \geq 3$. Then we obtain $u_p = \frac{2}{3}(2^{p-1} + (-1)^p)$ and $v_p = \frac{1}{3}(2^p + (-1)^{p+1})$ for $p \geq 2$.

By Lemma 19

$$\mu_{2.0}'(1,2p) = a^p_{XY} + a^p_{YZ} + a^p_{ZX} = 3v_p = 2^p + (-1)^{p+1},$$

$$\mu_{2.0}'(2,2p) = a^p_{XX} + a^p_{YZ} + a^p_{ZY} = u_p + 2v_p = 2^p \ ,$$

and $$\mu_{2.0}'(3,2p) = a^p_{XX} + a^p_{YY} + a^p_{ZZ} = 3u_p = 2^p + (-1)^p 2.$$

Since $\mu_2'(j,2p) = \mu_{2.0}'(j,2p) + \mu_{2.1}'(j,2p)$ and $\mu_{2.0}'(j,2p) = \mu_{2.1}'(j,2p)$ we obtain the announced results. $\square$

Remark 21 We see that $\mu_2'(j,2p) \simeq 2^{p+1}$ and this is to compare with the number $\mu_2(j,2p) = 2 \times 3^p$ of perfect matchings of type 2 in $FS(j,2p)$ (see backward in Section 2).

For a given graph $G = (V, E)$ a *strong matching* (or *induced matching*) is a matching S such that no two edges of S are joined by an edge of G. That is, S is the set of edges of the subgraph of G induced by the set $V(S)$. We consider cubic graphs having a perfect matching which is the union of two strong matchings that we call *Jaeger's graph* (in his thesis [9] Jaeger called these cubic graphs *equitable*). We call *Jaeger's matching* a perfect matching M of a cubic graph G which is the union of two strong matchings M_B and M_R. Set $B = V(M_B)$ (the blue vertices) and $R = V(M_R)$ (the red vertices). An edge of G is said *mixed* if its end vertices have distinct colours. Since the set of mixed edges is $E(G) \setminus M$, the 2-factor $G \setminus M$ is even and $\mid B \mid = \mid M \mid$. Thus, every Jaeger's graph G is a cubic 3-edge colourable graph and for any Jaeger's matching $M = M_B \cup M_R$, $\mid M_B \mid = \mid M_R \mid$. See, for instance, [3] and [4] for some properties of these graphs.

In this subsection we determine the values of j and k for which a graph $FS(j,k)$ is a Jaeger's graph.

Lemma 22 *If $G = FS(j,k)$ is a Jaeger's graph* (with $k \geq 3$) *and $M = M_B \cup M_R$ is a Jaeger's matching of G then M is a perfect matching of type* 1.

Proof Suppose that M is of type 2 and suppose without loss of generality that there are two edges of M between C_0 and C_1, for instance x_0x_1 and y_0y_1. Then $C_0 \cap M = \{t_0z_0\}$ and $C_1 \cap M = \{t_1z_1\}$. Suppose that x_0x_1 and y_0y_1 belong to M_B. Since M_B is a strong matching, t_0z_0 and t_1z_1 belong to $M \setminus M_B = M_R$. This is impossible because M_R is also a strong matching. By symmetry there are no two edges of M_R between C_0 and C_1. Then there is one edge of M_B between C_0 and C_1, x_0x_1 for instance, and one edge of M_R between C_0 and C_1, y_0y_1 for instance. Since M_B and M_R are strong matchings, there is no edge of M in $C_0 \cup C_1$, a contradiction. Thus, M is a perfect matching of type 1. $\square$

Lemma 23 *If $G = FS(j,k)$ is a Jaeger's graph* (with $k \geq 3$) *then either ($j = 1$ and $k \equiv 1$ or 2 $(mod\ 3)$) or ($j = 3$ and $k \equiv 0$ $(mod\ 3)$).*

Proof Let $M = M_B \cup M_R$ be a Jaeger's matching of G. By Lemma 22 M is a perfect matching of type 1. Suppose without loss of generality that $M_B \cap E(C_0) = \{x_0t_0\}$. Since M_B is a strong matching there is no edge of M_B between C_0 and C_1. Suppose, without loss of generality, that the edge in M_R joining C_0 to C_1 is y_0y_1. Consider the claws C_0, C_1 and C_2. Since M_B and M_R are strong matchings, we can see that the choices of $x_0t_0 \in M_B$ and $y_0y_1 \in M_R$ fixes the positions of the other edges of M_B and M_R. More precisely, $\{t_1z_1, y_2t_2\} \subset M_B$ and $\{x_1x_2, z_2z'_3\} \subset M_R$. This unique configuration is depicted in Figure 5.

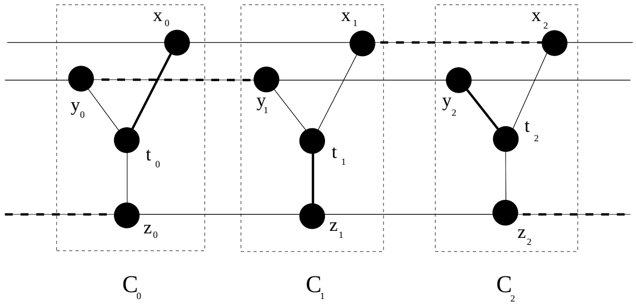

Fig. 5. Strong matchings M_B (bold edges) and M_R (dashed edges)

If $k \geq 4$ then we see that $z_2z_3 \in M_R$, $x_3t_3 \in M_B$, and $y_3y'_4 \in M_R$. So, the local situation in C_3 is similar to that in C_0, and we can see that there is a unique Jaeger's matching $M = M_B \cup M_R$ such that $x_0t_0 \in M_B$ and $y_0y_1 \in M_R$ in the graph $FS(j,k)$. We have to verify the coherence of the connections between the claws C_{k-1} and C_0. We note that $M_B = M \cap (\cup_{i=0}^{i=k-1} E(C_i))$ and M_R is a strong matching included in the 2-factor induced by $\cup_{i=0}^{i=k-1}\{V(C_i) \setminus \{t_i\}\}$.

Case 1: $k = 3p$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $y_{k-1}t_{k-1} \in M_B$, $x_{k-2}x_{k-1} \in M_R$ and $z'_{k-1}z_0 = z_{k-1}z'_0 \in M_R$ (that is, $z_{k-1}z_0 \in M_R$). Thus, $z_{k-1}z_0$, $y_{k-1}y_0$ and $x_{k-1}x_0$ are edges of $FS(j,3p)$ and we must have $j = 3$.

Case 2: $k = 3p+1$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $x_{k-1}t_{k-1} \in M_B$ (that is, $x_{k-1}x_0 \notin E(G)$), $z_{k-2}z_{k-1} \in M_R$ and $z'_{k-1}z_0 = y_{k-1}y'_0 \in M_R$ (that is, $y_{k-1}z_0 \in M_R$). Thus, $y_{k-1}z_0$, $x_{k-1}y_0$ and $z_{k-1}x_0$ are edges of $FS(j,3p+1)$ and we must have $j = 1$.

Case 3: $k = 3p+2$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $z_{k-1}t_{k-1} \in M_B$, $y_{k-2}y_{k-1} \in M_R$ and $z'_{k-1}z_0 = x_{k-1}x'_0 \in M_R$ (that is $x_{k-1}z_0 \in M_R$). Thus, $x_{k-1}z_0$, $y_{k-1}x_0$ and $z_{k-1}y_0$ are edges of $FS(j,3p+2)$ and we must have $j = 1$. □

Remark 24 It follows from Lemma 23 that for every $k \geq 3$ the graph $FS(2,k)$ is not a Jaeger's graph. This is obvious when k is odd, since the flower snarks have chromatic index 4.

Then, we obtain the following.

Theorem 25 *For $j \in \{1,2,3\}$ and $k \geq 2$, the graph $G = FS(j,k)$ is a Jaeger's graph if and only if* *either* $k \equiv 1$ or 2 $(mod\ 3)$ *and* $j = 1$,

or $k \equiv 0$ $(mod\ 3)$ *and* $j = 3$.

Moreover, $FS(1,2)$ *has* 3 *Jaeger's matchings and for* $k \geq 3$ *a Jaeger's graph* $G = FS(j,k)$ *has exactly* 6 *Jaeger's matchings.*

Proof For $k = 2$ we remark that $FS(1,2)$ (that is the cube) has exactly three distinct Jaeger's matchings M_1, M_2 and M_3. Following our notations: $M_1 = \{x_0t_0, t_1z_1\} \cup \{y_0y_1, z_0x_1\}$, $M_2 = \{z_0t_0, t_1y_1\} \cup \{y_0z_1, x_0x_1\}$ and $M_3 = \{y_0t_0, t_1x_1\} \cup \{z_0z_1, x_0y_1\}$.

For $k \geq 3$, by Lemma 23, condition

$(*)$ ($j = 1$ and $k \equiv 1$ or 2 $(mod\ 3)$) or ($j = 3$ and $k \equiv 0$ $(mod\ 3)$)

is a necessary condition for $FS(j,k)$ to be a Jaeger's graph.

Consider the function $\Phi_{X,Y} : V(G) \rightarrow V(G)$ such that for every i in $\mathbf{Z}_k$, $\Phi_{X,Y}(t_i) = t_i$, $\Phi_{X,Y}(z_i) = z_i$, $\Phi_{X,Y}(x_i) = y_i$ and $\Phi_{X,Y}(y_i) = x_i$. Define similarly $\Phi_{X,Z}$ and $\Phi_{Y,Z}$. For $j = 1$ or 3 these functions are automorphisms of $FS(j,k)$. Thus, the process described in the proof of Lemma 23 is a constructive process of all Jaeger's matchings in a graph $FS(j,k)$ (with $k \geq 3$) verifying condition (*).

We remark that for any choice of an edge e of C_0 to be in M_B there are two distinct possible choices for an edge f between C_0 and C_1 to be in M_R, and such a pair $\{e, f\}$ corresponds exactly to one Jaeger's matching. Then, a Jaeger's graph $FS(j,k)$ (with $k \geq 3$) has exactly 6 Jaeger's matchings. □

Remark 26 The *Berge-Fulkerson Conjecture* states that if G is a bridgeless cubic graph, then there exist six perfect matchings $M_1, \ldots, M_6$ of G (not necessarily distinct) with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$ (this conjecture is attributed to Berge in [16] but appears in [5]). Using each colour of a cubic 3-edge colourable graph twice, we see that such a graph verifies the Berge-Fulkerson Conjecture. Very few is known about this conjecture except that it holds for the Petersen graph and for cubic 3-edge colourable graphs. So, Berge-Fulkerson Conjecture holds for Jaeger's graphs, but generally we do not know if we can find six distinct perfect matchings. We remark that if $FS(j,k)$, with $k \geq 3$, is a Jaeger's graph then its six Jaeger's matchings are such that every edge is contained in exactly two of them.

5 2-factor hamiltonian cubic graphs

Recall that a simple graph of maximum degree $d > 1$ with edge chromatic number equal to d is said to be a *Class* 1 *graph*. For any d-regular simple graph (with $d > 1$) of even order and of Class 1, for any minimum edge-colouring of such a graph, the set of edges having a given colour is a perfect matching (or 1-factor). Such a regular graph is also called a 1-*factorable graph*. A Class 1 d-regular graph of even order is *strongly hamiltonian* or *perfectly* 1-*factorable* (or is a *Hamilton graph* in the Kotzig's terminology [10]) if it has an edge colouring such that the union of any two colours is a hamiltonian cycle. Such an edge colouring is said to be *a Hamilton decomposition* in the Kotzig's terminology. In [11] by using two operations ρ and π (described also in [10]) and starting from the θ-graph (two vertices joined by three parallel edges) he obtains all strongly hamiltonian cubic graphs, but these operations do not always preserve planarity. In his paper [10] he describes a method for constructing planar strongly hamiltonian cubic graphs and he deals with the relation between strongly hamiltonian cubic graphs and 4-regular graphs which can be decomposed into two hamiltonian cycles. See also [12] and a recent work on strongly hamiltonian cubic graphs [2] in which the authors give a new construction of strongly hamiltonian graphs.

A Class 1 regular graph such that every edge colouring is a Hamilton decomposition is called a *pure Hamilton graph* by Kotzig [10]. Note that K_4 is a pure Hamilton graph and every cubic graph obtained from K_4 by a sequence of triangular extensions is also a pure Hamilton cubic graph. In the paper [10] of Kotzig, a consequence of his Theorem 9 (p.77) concerning pure Hamilton graphs is that the family of pure Hamilton graphs that he exhibits is precisely the family obtained from K_4 by triangular extensions. Are there others pure Hamilton cubic graphs ? The answer is "yes".

We remark that 2-factor hamiltonian cubic graphs defined above (see Definition 16) are pure Hamilton graphs (in the Kotzig's sense) but the converse is false because K_4 is 2-factor hamiltonian and the pure Hamilton cubic graph on 6 vertices obtained from K_4 by a triangular extension (denoted by PR_3) is not 2-factor hamiltonian. Observe that the operation of triangular extension preserves the property "pure Hamilton", but does not preserve the property "2-factor hamiltonian". The Heawood graph H_0 (on 14 vertices) is pure Hamiltonian, more precisely it is 2-factor hamiltonian (see [7] Proposition 1.1 and Remark 2.7). Then, the graphs obtained from the Heawood graph H_0 by triangular extensions are also pure Hamilton graphs.

A *minimally* 1-*factorable* graph G is defined by Labbate and Funk [7] as a Class 1 regular graph of even order such that every perfect matching of G is contained in exactly one 1-factorization of G. In their article they study bipartite minimally 1-factorable graphs and prove that such a graph G has necessarily a degree $d \leq 3$. If G is a minimally 1-factorable cubic graph then the complementary 2-factor of any perfect matching has a unique decomposition into two perfect matchings, therefore this 2-factor is a hamiltonian cycle of G, that is G is 2-factor hamiltonian. Conversely it is easy to see that any 2-factor hamiltonian cubic graph is minimally 1-factorable. The complete bipartite graph $K_{3,3}$ and the Heawood graph H_0 are examples of 2-factor hamiltonian bipartite graph given by Labbate and Funk. Starting from H_0, from $K_{1,3}$ and from three copies of any tree of maximum degree 3 and using three operations called *amalgamations* the authors exhibit an infinite family of bipartite 2-factor hamiltonian cubic graphs, namely the $poly - HB - R - R^2$ graphs (see [7] for more details). Except H_0, these graphs are exactly cyclically 3-edge connected. Others structural results about 2-factor hamiltonian bipartite cubic graph are obtained in [13], [14]. These results have been completed and a simple method to generate 2-factor hamiltonian bipartite cubic graphs was given in [6].

Proposition 27 (Lemma 3.3, [6]) *Let G be a 2-factor hamiltonian bipartite cubic graph. Then G is 3-connected and $\mid V(G) \mid \equiv 2 \ (mod\ 4)$.*

Let G_1 and G_2 be disjoint cubic graphs, $x \in v(G_1)$, $y \in v(G_2)$. Let x_1, x_2, x_3 (respectively y_1, y_2, y_3) be the neighbours of x in G_1 (respectively, of y in G_2). The cubic graph G such that $V(G) = (V(G_1)\backslash\{x\})\cup(V(G_2)\backslash\{y\})$ and $E(G) = (E(G_1)\backslash\{x_1x, x_2x, x_3x\})\cup(E(G_2)\backslash\{y_1y, y_2y, y_3y\})\cup\{x_1y_1, x_2y_2, x_3y_3\}$ is said to be a *star product* and G is denoted by $(G_1, x)*(G_2, y)$. Since $\{x_1y_1, x_2y_2, x_3y_3\}$ is a cyclic edge-cut of G, a star product of two 3-connected cubic graphs has cyclic edge-connectivity 3.

Proposition 28 (Proposition 3.1, [6]) *If a bipartite cubic graph G can be represented as a star product $G = (G_1, x)*(G_2, y)$, then G is 2-factor hamiltonian if and only if G_1 and G_2 are 2-factor hamiltonian.*

Then, taking iterated star products of $K_{3,3}$ and the Heawood graph H_0 an infinite family of 2-factor hamiltonian cubic graphs is obtained. These graphs (except $K_{3,3}$ and H_0) are exactly cyclically 3-edge connected. In [6] the authors conjecture that the process is complete.

Conjecture 29 *(*Funk, Jackson, Labbate, Sheehan (2003)[6]*) Let G be a bipartite 2-factor hamiltonian cubic graph. Then G can be obtained from $K_{3,3}$ and the Heawood graph H_0 by repeated star products.*

The authors precise that a smallest counterexample to Conjecture 29 is a cyclically 4-edge connected cubic graph of girth at least 6, and that to show this result it would suffice to prove that H_0 is the only 2-factor hamiltonian cyclically 4-edge connected bipartite cubic graph of girth at least 6. Note that some results have been generalized in [1].

To conclude, we may ask what happens for non bipartite 2-factor hamiltonian cubic graphs. Recall that K_4 and $FS(1,3)$ (the "Triplex Graph" of Robertson, Seymour and Thomas [15]) are 2-factor hamiltonian cubic graphs. By Corollary 17 the graphs $FS(j,k)$ with k odd and $j=1$ or 3 introduced in this paper form a new infinite family of non bipartite 2-factor hamiltonian cubic graphs. We remark that they are cyclically 6-edge connected. Can we generate others families of non bipartite 2-factor hamiltonian cubic graphs ? Since PR_3 (the cubic graph on 6 vertices obtained from K_4 by a triangular extension) is not 2-factor hamiltonian and $PR_3 = K_4 * K_4$, the star product operation is surely not a possible tool.

References

[1] M. Abreu, A.A. Diwan, Bill Jackson, D. Labbate, and J. Sheehan. Pseudo 2-factor isomorphic regular bipartite graphs. *J. Combin. Theory Ser. B*, 98:432–442, 2008.

[2] S. Bonvicini and G. Mazzuoccolo. On perfectly one-factorable cubic graphs. *Electronic Notes in Discrete Math.*, 24:47–51, 2006.

[3] J.L. Fouquet, H. Thuillier, J.M. Vanherpe, and A.P. Wojda. On linear arboricity of cubic graphs. *LIFO Univ. d'Orléans - Research Report*, 2007-13:1–28, 2007.

[4] J.L. Fouquet, H. Thuillier, J.M. Vanherpe, and A.P. Wojda. On isomorphic linear partition in cubic graphs. *Discrete Math., to appear*, 2008.

[5] D.R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, 1:168–194, 1971.

[6] M. Funk, Bill Jackson, D. Labbate, and J. Sheehan. 2-factor hamiltonian graphs. *J. Combin. Theory Ser. B*, 87:138–144, 2003.

[7] M. Funk and D. Labbate. On minimally one-factorable r-regular bipartite graphs. *Discrete Math.*, 216:121–137, 2000.

[8] R. Isaacs. Infinite families of non-trivial trivalent graphs which are not Tait colorable. *Am. Math. Monthly*, 82:221–239, 1975.

[9] F. Jaeger. Etude de quelques invariants et problèmes d'existence en théorie de graphes. Thèse d'état, IMAG, Grenoble, 1976.

[10] A. Kotzig. Balanced colourings and the four colour conjecture. In *Proc. Sympos. Smolenice, 1963.* Publ. House Czechoslovak Acad. Sci., Prague, 63-82, 1964.

[11] A. Kotzig. Construction for hamiltonian graphs of degree three (in russian). *Čas. pěst. mat.*, 87:148–168, 1962.

[12] A. Kotzig and J. Labelle. Quelques problèmes ouverts concernant les graphes fortement hamiltoniens. *Ann. Sci. Math. Québec,* 3:95-106, 1979.

[13] D. Labbate. On 3-cut reductions of minimally 1-factorable cubic bigraphs. *Discrete Math.*, 231:303–310, 2001.

[14] D. Labbate. Characterizing minimally 1-factorable r-regular bipartite graphs. *Discrete Math.*, 248:109–123, 2002.

[15] N. Robertson and P. Seymour. Excluded minor in cubic graphs. *(announced)*, see also www.math.gatech.edu/˜ thomas/OLDFTP/cubic/graphs.

[16] P. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.(3)*, 38:423–460, 1979.