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Functional Term Rewriting Systems

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Abstract. This research report proposes the theoretical foundations of a new formal tool for symbolic verification of finite systems. Some approaches reduce the problem of system verification to the reachability problem in term rewriting systems (TRSs). In our approach, states are encoded by terms in a BDD-like manner and the transition relation is represented by a new rewriting relation so called Functional Term Rewriting Systems (FTRSs). First, we show that FTRSs are as expressive as TRSs. Then, we focus on a subclass of FTRSs, so called Elementary Functional Term Rewriting Systems (EFTRSs), and we show that EFTRSs preserve the FTRSs expressiveness. The main advantage of EFTRSs is that they are well adapted for acceleration techniques usually used in saturation algorithms on BDD-like data structures. Our experiments show that for well-known protocols (e.g. Tree Arbiter, Percolate, Round Robin Mutex protocols,...) our tool is not only better than other rewriting tools such as Timbuk or Maude, but also competitive with other model-checkers such as SPIN, NuSMV or SMART. Moreover, it can also be applied to model-checking invariant properties which are a particular subclass of linear temporal logic formula (LTL).

1 Introduction

In the earliest 90s, in order to show their product (complex electronic components) secure, electronics industry integrates some techniques coming from the world of formal methods: the Binary Decision Diagrams (BDDs) [6, 7]. BDDs are structures coding boolean functions.

They can be considered as trees where nodes specify the choice of a value for the variable corresponding to this node. A total ordered relation on variables ensures the existence of a canonical BDD representation of a given boolean function. Some reduction techniques and data structures well adapted for data sharing leads to very efficient implementation in practice [42, 36]. Considering a state or a configuration of a system as a BDD and the transition relation as operations on BDDs, the techniques mentioned above have made the exhaustive exploration, verification of huge systems (sometimes composed of more than 1 Billion states [42, 36]) possible.
Moreover, as shown in [8], the BDDs are expressive enough for handling a large class of finite systems. In [38], these structures have even been shown to be adapted for some classes of dynamic systems.

The key point of these data structures being the number of variables, some people work on BDD-like data structures [43, 12]. Some others have worked in order to make the application domain of BDDs larger: [3, 34, 33, 39, 44, 40].

In this paper, we aim at proposing a technique mixing acceleration techniques commonly used in the BDD world for saturation [14], and rewriting techniques. Indeed, given a system, its configurations are encoded by terms and the transition relation is described by a functional term rewriting system (FTRS). FTRSs are shown to be as expressive as term rewriting systems (TRSs). We also focus on a subclass of FTRSs so called elementary functional term rewriting systems (EFTRSs). EFTRSs preserve the expressiveness of FTRS and are well adapted to acceleration techniques.

The remainder of the paper is structured as follows. Section 2 recalls the background on terms, TRSs and rewriting. Section 3 illustrates how systems are specified by terms and how the verification of systems can be performed using rewriting techniques. Section 4 and 5 introduce FTRS and EFTRS, and present our main results concerning their expressiveness, i.e., FTRS and EFTRS are as expressive as TRS. Finally, Section 6 shows how the system described in Section 3 is specified in terms of EFTRS. This section also describes how model-checking can be performed using FTRS/EFTRS and promising results in the field of state space exploration and model-checking are exhibited. For well-known protocols (e.g. Tree Arbiter, Percolate, Round Robin Mutex protocols,...) our tool is not only better than other rewriting tools such as Timbuk [31, 27] or Maude [19], but also competitive with other model-checkers such as SPIN [35], NuSMV [17] or SMART [10].

### 2 Preliminaries on Terms and TRSs

Comprehensive surveys can be found in [24, 2] for term rewriting systems.

For now, we focus on terms composed of binary functional symbols and of the constant $\bot$. We denote by $T_{\text{bin}}$ this set of symbols. Let $X$ be a countable set of variables. $T(T_{\text{bin}}, X)$ denotes the set of terms built on those symbols and containing variables, and $T(T_{\text{bin}})$ denotes the set of ground terms (terms without variables). A substitution is a function $\sigma$ from $X$ into $T(T_{\text{bin}}, X)$, which can be extended uniquely to an endomorphism of $T(T_{\text{bin}}, X)$. A position $p$ for a term $t$ is a word over $N$. The empty sequence $\epsilon$ denotes the top-most position. The set $\text{Pos}(t)$ of positions of a term $t$ is inductively defined by $\text{Pos}(t) = \{ \epsilon \} \cup \{ 1.p \mid p \in \text{Pos}(t_1) \} \cup \{ 2.p \mid p \in \text{Pos}(t_2) \}$, and $\text{Pos}(t) = \{ \epsilon \}$ when $t \in X$ or $t = \bot$. Let $\preceq$ be a partial order on positions. Let $p_1$ and $p_2$ be two words of $N^*$, we say that $p_1 \preceq p_2$ if there exists $\omega \in N^*$ such that $p_1 = p_2.\omega$.

If $p \in \text{Pos}(t)$, then $t|_p$ denotes the subterm of $t$ at position $p$ and $t[s]_p$ denotes the term obtained by replacement of the subterm $t|_p$ at position $p$ by the term $s$. We also denote by $t(p)$ the symbol occurring in $t$ at position $p$. Given a term
t ∈ T(F_{bin}, X) and A a set of symbols, let Pos_A(t) = {p ∈ Pos(t) | t(p) ∈ A}. Thus Pos(t) is the set of positions of t, at each of which a function symbol appears. The set of frontier positions of a term t, denoted by FPos(t), is defined as follows: FPos(t) = \{ p ∈ Pos(t) | p.1 /∈ Pos(t) \}. The set of variables occurring a term t ∈ T(F, X) is denoted by Var(t). More formally, Var(t) = \{ t(p) | p ∈ Pos_X(t) \}.

A TRS R is a set of rewrite rules (l, r) ∈ T(F_{bin}, X) × T(F_{bin}, X), also denoted by l → r where Var(r) ⊆ Var(l) and l /∈ X. The TRS R induces a rewriting relation →^R on terms whose reflexive transitive closure is written →^* R. More precisely, we say that t →^R t′ if there exist a position p of Pos(t), a rule l → r ∈ R and a substitution σ : X ↦→ T(F) such that t|_p = lσ and t′ = t[rσ]_p.

The set of R-descendants of a set of terms E ⊆ T(F_{bin}) is R^*(E) = \{ s ∈ T(F_{bin}) | t →^* R s ∧ t ∈ E \}.

3 Verification of Systems using Rewriting

This section shows how the tree arbiter protocol (TAP), a protocol solving mutual exclusion problem for accessing to a shared resource, can be formalised in terms and TRSs. We also describe how a safety property concerning this protocol can be formulated in a reachability problem.

3.1 Presentation of the Tree Arbiter Protocol (TAP)

As mentioned above, the TAP is an asynchronous circuit that solves the mutual exclusion problem by building a tree of arbiter cells. This circuit has been introduced by Dill [25] and has been handled as a problem of asynchronous model-checking in [18]. Recently, a version of the TAP has been tackled using a technique of tree regular model-checking in [1].

The circuit works by performing elimination rounds: an arbiter cell arbitrates between its two children. The leaves of the tree are processors, which may want to access asynchronously a shared resource. The N processors at the lowest level are arbitrated by N/2 cells. The winners of that level are arbitrated by the next level, and so forth.

3.2 Specification of the TAP in Terms and TRS

We use the binary symbols i, r, t and b respectively for the specification of states idle, requesting, token and token below where:

- state idle means that all children of a cell (or a processor) do(es) not do anything.
- state requesting means that this processor (or one of children of this cell) wants to access the shared resource.
- state token means that this processor (or this cell) has been granted the shared resource.
state token below means that the shared resource (or the token) is somewhere in one of subtree below this node.

Given an initial total idle state for a model of 4 processors \( u = t(i(i(⊥, ⊥), i(⊥, ⊥)), i(i(⊥, ⊥), i(⊥, ⊥))) \) where all processors are idle and the root cell holds the token (See Figure 1a).

Requests are propagated upwards until the root, holding the token, is reached:

\[
i(x, r(y, z)) \rightarrow r(x, r(y, z)) \quad (1)
\]
\[
i(r(x, y), z) \rightarrow r(r(x, y), z) \quad (2)
\]
\[
i(⊥, ⊥) \rightarrow r(⊥, ⊥) \quad (3)
\]

By using only these requests rules, the system may reach a total requesting state \( v = t(r(r(⊥, ⊥), r(⊥, ⊥)), r(r(⊥, ⊥), r(⊥, ⊥))) \) (See Figure 1b).

The root cell grants the resource to at most one child, and the grant propagates downward to one of the processors. If both children of a cell are requesting the resource, then the cell chooses one of them non-deterministically:

\[
t(x, r(y, z)) \rightarrow b(x, t(y, z)) \quad (4)
\]
\[
t(r(x, y), z) \rightarrow b(t(x, y), z) \quad (5)
\]

Consequently, in such a way, only a processor having requested the resource will get it. When the processor is done with the resource, it sends a release request that is propagated upwards.

\[
b(x, t(⊥, ⊥)) \rightarrow t(x, i(⊥, ⊥)) \quad (6)
\]
\[
b(t(⊥, ⊥), z) \rightarrow t(i(⊥, ⊥), z) \quad (7)
\]

Along the come-back to the root process of the token, three behaviours are possible for a cell:

- If all of the children of such a cell are in the idle state then this cell gives the token to the next higher neighboured cell and goes also to the idle state.

\[
b(x, t(i(y, z), i(u, v))) \rightarrow t(x, i(i(y, z), i(u, v))) \quad (8)
\]
\[
b(t(i(x, y), i(u, v)), z) \rightarrow t(i(x, y), i(u, v)), z) \quad (9)
\]

- If one of the children of such a cell has requested the resource then:
  - either the next state of the cell will be the request one and the token goes to the next higher neighboured cell using following rules

\[
b(x, t(r(y, z), i(u, v))) \rightarrow t(x, r(r(y, z), i(u, v))) \quad (10)
\]
\[
b(x, t(r(y, z), r(u, v))) \rightarrow t(x, r(i(y, z), r(u, v))) \quad (11)
\]
\[
b(r(x, y), i(u, v)), z) \rightarrow t(r(r(x, y), i(u, v)), z) \quad (12)
\]
\[
b(t(i(x, y), r(u, v)), z) \rightarrow t(r(i(x, y), r(u, v)), z) \quad (13)
\]

  - or the next state of the cell will be the token below one because the cell has chosen to give the token to one of its requesting children using rules (4) or (5).
3.3 Verification using Reachability Analysis

An interesting safety property $\phi$ to check on the TAP is that there is always only one token circulating in the hierarchy tree. So, one can formulated this property as a predicate to check on each reachable configuration.

Considering the symbols $\top$ and $\bot$ as respectively the true and false values, let $\text{OneToken}$ be the predicate defined as follows for a term $\alpha \in \mathcal{T}(\mathcal{F}_{\text{bin}})$:

$$\text{OneToken}(\alpha) = \begin{cases} \top & \text{if } \forall p, p' \text{Pos}(\alpha), \\ \alpha(p) = \alpha(p') = t \Rightarrow p = p' \\ \bot & \text{otherwise.} \end{cases}$$

Consequently,

$$\text{TAP satisfies } \phi \iff \forall \alpha \in \mathcal{R}^*\{u\}, \text{OneToken} = \top,$$

with $u$ is the term $t(i(t(\bot, \bot), i(\bot, \bot)), i(t(\bot, \bot), i(\bot, \bot)))$.

![Fig. 1: a) total idle global state $u$ and b) total requesting global state $v$](image)

4 Functional Term Rewriting Systems

In this section, we propose a new formalism, so called Functional Term Rewriting Systems (FTRSs) being able to simulate classical TRS. The motivations of this new formalism are on the one hand that to make the control of the rewriting process easier and completely independent of the implementation of the rewriting engine, and on the other hand to offer a formalism well adapted to acceleration techniques commonly used in saturation algorithm on BDDs.

We first introduce new functional symbols so called non-terminals. We denote by $F_{NT}$ the set of non-terminal symbols such that $F_{NT} \cap F_{\text{bin}} = \emptyset$ and for each $F \in F_{NT}$, the arity of $F$ is one. To make the distinction between functional symbols easier, from now, only functional symbols of $F_{NT}$ will be written in capital letters. A functional TRS $\mathcal{R}_{\lambda}$ is a set of rules of the form $F(t) \rightarrow \alpha$ where $F \in F_{NT}$, $t \in \mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X})$ and $\alpha \in \mathcal{T}(\mathcal{F}_{\text{bin}} \cup F_{NT}, \mathcal{X})$. A FTRS $\mathcal{R}_{\lambda}$ induces a functional rewriting relation $\rightarrow_{\mathcal{R}_{\lambda}}$ on terms in $\mathcal{T}(\mathcal{F}_{\text{bin}} \cup F_{NT})$: $u \rightarrow_{\mathcal{R}_{\lambda}} v$ iff there exist a position $p$ of $u$ and a rule $F(t) \rightarrow \alpha$ in $\mathcal{R}_{\lambda}$ such that there exists a substitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}})$ with $u|_p = F(t)\sigma$ and $v = u[\alpha \sigma]_p$. 
We also denote by $\rightarrow^*_{\mathcal{R}_\lambda}$ the transitive closure of the functional rewriting relation $\rightarrow_{\mathcal{R}_\lambda}$. The following definition describes the set of terms of $\mathcal{T}(\mathcal{F}_{bin})$ reachable by function rewriting from a term of $\mathcal{T}(\mathcal{F}_{bin} \cup \mathcal{F}_{NT})$.

**Definition 1 ($\mathcal{R}_\lambda^*(E)$).** Let $\mathcal{R}_\lambda$ be a FTRS and $E \subseteq \mathcal{T}(\mathcal{F}_{bin} \cup \mathcal{F}_{NT})$ such that $E \cap \mathcal{T}(\mathcal{F}_{bin}) = \emptyset$. The set of reachable terms of $\mathcal{T}(\mathcal{F}_{bin})$ from $E$ is denoted by $\mathcal{R}_\lambda^*(E)$ and is defined as follows:

$$\mathcal{R}_\lambda^*(E) = \{ \beta \in \mathcal{T}(\mathcal{F}_{bin})| \alpha \rightarrow^*_{\mathcal{R}_\lambda} \beta \land \alpha \in E \}.$$ 

Now, let us compare the expressiveness of FTRSs and TRSs. Proposition 1 shows that a rewrite step using $\mathcal{R}$ can be simulated with a FTRS $\mathcal{R}_\lambda$.

**Proposition 1.** Let $\mathcal{R}$ be a TRS over $\mathcal{T}(\mathcal{F}_{bin}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}_{bin}, \mathcal{X})$, $\alpha \in \mathcal{T}(\mathcal{F}_{bin})$, $\beta \in \mathcal{T}(\mathcal{F}_{bin})$ and $l \rightarrow r \in \mathcal{R}$. Let $\mathcal{R}_\lambda$ be the FTRS such that $\mathcal{R}_\lambda = \{ F_{l \rightarrow r}(a(x, y)) \rightarrow a(F_{l \rightarrow r}(a(x, y)), F_{l \rightarrow r}(a(x, y))) | a \in \mathcal{F}_{bin} \cup \{ F_{l \rightarrow r}(l) \rightarrow r \} \}$. Thus, $\alpha \rightarrow_{\{l \rightarrow r\}} \beta \iff F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \beta$.

**Proof.** $\alpha \rightarrow_{\{l \rightarrow r\}} \beta \Rightarrow F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \beta$: By Definition, there exist a position $p$ of $\alpha$ and a substitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{bin})$ such that $|\alpha|_p = l\sigma$ and $\beta = \alpha[r\sigma]_p$. Let $\alpha'$ be the term constructed as follows: $\alpha' = \alpha[F_{l \rightarrow r}(\alpha|_p)]_p$. Clearly, using only rules of the form $F_{l \rightarrow r}(a(x, y)) \rightarrow a(F_{l \rightarrow r}(x), y)$ and $F_{l \rightarrow r}(a(x, y)) \rightarrow a(x, F_{l \rightarrow r}(y))$, one has

$$F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \alpha'.$$

By hypothesis, there exist a substitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{bin})$ such that $|\alpha|_p = l\sigma$. Consequently, one has $F_{l \rightarrow r}(\alpha|_p) = F_{l \rightarrow r}(l\sigma)$. Thus, $F_{l \rightarrow r}(\alpha|_p) = F_{l \rightarrow r}(l\sigma)$. By construction of $\alpha'$, there exist a rule of $\mathcal{R}_\lambda$, i.e. $F_{l \rightarrow r}(l) \rightarrow r$, a substitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{bin})$ and a position $p$ such that $\alpha'|_p = F_{l \rightarrow r}(l)\sigma$. Consequently, one can construct the term $\alpha'[r\sigma]$ which is equal to $\beta$ by construction of $\alpha'$. So, one has

$$\alpha' \rightarrow_{\mathcal{R}_\lambda} \beta.$$

By (14) and (15), one can deduce that $F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \beta$. Consequently, $\alpha \rightarrow_{\{l \rightarrow r\}} \beta \Rightarrow F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \beta$.

- $\alpha \rightarrow_{\{l \rightarrow r\}} \beta \Leftarrow F_{l \rightarrow r}(\alpha) \rightarrow^*_{\mathcal{R}_\lambda} \beta$: Let $\mathcal{R}_\lambda'$ be the FTRS such that $\mathcal{R}_\lambda = \mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r | l \rightarrow r \in \mathcal{R} \}$. Trivially, one can show that $\mathcal{R}_\lambda^*(\{ F_{l \rightarrow r}(\alpha) \}) = \emptyset$. Consequently, it implies that, for obtaining the term $\beta \in \mathcal{R}_\lambda^*(\{ F_{l \rightarrow r}(\alpha) \})$, the rule $F_{l \rightarrow r}(\alpha) \rightarrow_{\mathcal{R}_\lambda} \beta$ to be applied at least one time. Actually, we claim that this rule is applied exactly one times. Indeed, let us proceed by induction on the rewriting path leading us from $F_{l \rightarrow r}(\alpha)$ to $\beta$, i.e. there exist $t_0, \ldots, t_n \in \mathcal{T}(\mathcal{F}_{bin} \cup \mathcal{F}_{NT})$ such that $t_0 = F_{l \rightarrow r}(\alpha) \rightarrow_{\mathcal{R}_\lambda} t_1 \rightarrow_{\mathcal{R}_\lambda} \ldots \rightarrow_{\mathcal{R}_\lambda} t_n = \beta$. We first show that each $t_i$ has one position $p$ at most such that $t_i(p) = F_{l \rightarrow r}$, Let $p_n$ be the following proposition: for all $p, p' \in \mathcal{Pos}(t_n)$, if $t_n(p) = t_n(p') = F_{l \rightarrow r}$ then $p = p'$. 

• \( P_0 \): Since \( t_0 = F_{l \rightarrow r}(\alpha) \) and \( \alpha \in \mathcal{T}(\mathcal{F}_{\text{bin}}) \), \( \varepsilon \) is the unique position of \( t_0 \) where \( F_{l \rightarrow r} \) occurs. So, \( P_0 \) is true.

• \( P_n \Rightarrow P_{n+1} \): Suppose \( P_n \) to be true. Consequently, \( t_n \) satisfies the property: \( p, p' \in \mathcal{P}os(t_n) \), if \( t_n(p) = t_n(p') = F_{l \rightarrow r} \). Moreover, since \( t_n \rightarrow_{\mathcal{R}_\lambda} t_{n+1} \), there exists necessarily a unique position \( p \) of \( t_n \) such that \( t_n(p) = F_{l \rightarrow r} \). Let us proceed by rewriting case analysis:
  
  * \( F_{l \rightarrow r}(a(x, y)) \rightarrow a(F_{l \rightarrow r}(x), y) \) is applied at position \( p \): So there exists a substitution \( \sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}}) \) such that \( t_n|_p = F_{l \rightarrow r}(a(x, y)) \sigma = F_{l \rightarrow r}(a(\sigma(x), \sigma(y))) \). Consequently, \( t_{n+1} = t_n[a(F_{l \rightarrow r}(x), y)\sigma] = t_n[a(F_{l \rightarrow r}(\sigma(x)), \sigma(y))] \). Since \( \sigma(x), \sigma(y) \in \mathcal{T}(\mathcal{F}_{\text{bin}}) \) and \( \forall p' \in \mathcal{P}os(t_n) \) such that \( p \neq p' \), \( t_n(p') \notin \mathcal{F}_{\text{NT}} \), \( t_{n+1} \) satisfies also \( \forall p, p' \in \mathcal{P}os(t_{n+1}) \), if \( t_{n+1}(p) = t_{n+1}(p') = F_{l \rightarrow r} \). More precisely, the unique position \( p' \) of \( t_{n+1} \) such that \( t_{n+1}(p) = F_{l \rightarrow r} \) is \( p' = p.1 \).
  
  * \( F_{l \rightarrow r}(a(x, y)) \rightarrow a(x), F_{l \rightarrow r}(y) \) is applied at position \( p \): the proof case is similar to the one above.

  * \( F_{l \rightarrow r}(l) \rightarrow r \) is applied at position \( p \): So there exists a substitution \( \sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}}) \) such that \( t_n|_p = F_{l \rightarrow r}(l) \sigma = F_{l \rightarrow r}(\sigma) \). Consequently, \( t_{n+1} = t_n[\sigma \sigma] \). Since \( r \in \mathcal{T}(\mathcal{F}_{\text{bin}}) \) and \( \sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}}) \), \( \sigma \sigma \) is thus a term of \( \mathcal{T}(\mathcal{F}_{\text{bin}}) \) and \( t_{n+1} \in \mathcal{T}(\mathcal{F}_{\text{bin}}) \). So, \( t_{n+1} \) satisfies the property: \( \forall p, p' \in \mathcal{P}os(t_{n+1}) \), if \( t_{n+1}(p) = t_{n+1}(p') = F_{l \rightarrow r} \).

We have just shown that each term within the rewriting path contains exactly one symbol \( F_{l \rightarrow r} \) and as soon as the rule \( F_{l \rightarrow r}(l) \rightarrow r \) is applied, a term of \( \mathcal{T}(\mathcal{F}_{\text{bin}}) \) is obtained. So the FTRS \( \mathcal{R}_\lambda \) cannot be applied anymore. Proving the claim.

So, the rewriting path leading us from \( F_{l \rightarrow r}(\alpha) \) to \( \beta \) is of the following form: \( t_0 = F_{l \rightarrow r}(\alpha) \rightarrow_{\mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r \}} t_1 \rightarrow_{\mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r \}} \ldots t_{n-1} \rightarrow_{\mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r \}} t_n = \beta \). With an induction very close to the previous one, one can show that there exist a position \( p \) of \( \alpha \) and a substitution \( \sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}}) \) such that \( t_{n-1} = \alpha[F(l)\sigma]|_p \). One can show also that \( t_{n-1}|_{p+1} = \alpha[\sigma]|_{p+1} \). Consequently,

\[
l\sigma = \alpha|_p. \tag{16}\]

Moreover, \( t_n = \beta = t_{n-1}[\sigma \sigma]_p \). Thus,

\[
\beta = \alpha[\sigma \sigma]_p. \tag{17}\]

By construction of \( \mathcal{R}_\lambda \), \( l \rightarrow r \in \mathcal{R} \). So, according to (16) and (17), there exist a rule \( l \rightarrow r \in \mathcal{R} \), a position \( p \in \mathcal{P}os(\alpha) \) and a substitution \( \sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}_{\text{bin}}) \) such that \( l\sigma = \alpha|_p \) and \( \beta = \alpha[\sigma \sigma]_p \). Consequently, \( \alpha \rightarrow_{\mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r \}} \beta \). To summarize, we have shown that \( \alpha \rightarrow_{\mathcal{R}_\lambda \setminus \{ F_{l \rightarrow r}(l) \rightarrow r \}} \beta \), concluding the proof.

Thus, the main result of this section is given in Theorem 1 i.e. for any TRS \( \mathcal{R} \), one can build a FTRS \( \mathcal{R}_\lambda \) simulating \( \mathcal{R} \).

**Theorem 1.** Let \( \mathcal{R} \) be a TRS over \( \mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X}) \), \( E \subseteq \mathcal{T}(\mathcal{F}_{\text{bin}}) \). Let \( l \rightarrow r \) be a rule of \( \mathcal{R} \) and \( \mathcal{R}_{l \rightarrow r} \) be the FTRS such that \( \mathcal{R}_{l \rightarrow r} = \{ F_{l \rightarrow r}(a(x, y)) \rightarrow \}

\[
\begin{align*}
\text{...}
\end{align*}
\]
with \( E' = \{G(t) | t \in E\} \).

**Proof.** \(- \mathcal{R}^*(E) \subseteq \mathcal{R}^*_\alpha(E')\): Let \( \alpha \in E \) and \( \beta \in \mathcal{R}^*\{\{\alpha\}\} \). By definition of \( \mathcal{R}^* \), there exist \( t_1 \rightarrow r_1, \ldots, t_n \rightarrow r_n \in \mathcal{R} \) and \( t_0, \ldots, t_n \), such that \( \alpha \rightarrow_{(t_0, t_n)} t_0 \rightarrow (t_1, t_n) \cdots \rightarrow (t_{n-1}, t_n) t_n = \beta \). The FTRS \( \mathcal{R}_{t_0 \rightarrow r_n} \) fulfills the expected conditions specified in Proposition 1. Consequently, according to Proposition 1, one has \( t_n \rightarrow_{(t_0, t_n)} t_1 \). Unfortunately, \( F_{t_0 \rightarrow r_n}(t_0) \notin E' \). However, \( t_0 = \alpha \) and \( \alpha \in E \). So, \( G(t_0) \in E' \). Trivially, \( G(t_0) \rightarrow \mathcal{G}(x) \rightarrow G(\mathcal{F}_{t_0 \rightarrow r_n}(\alpha)) \). Consequently, \( G(t_0) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_1) \rightarrow \mathcal{G}(x) \rightarrow t_1 \) implies that \( t_1 \in \mathcal{R}^*_\alpha(E') \). By generalizing the process, one obtains the following rewriting path: \( G(t_0) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_1) \rightarrow \mathcal{G}(x) \rightarrow t_1 \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_n) \rightarrow \mathcal{G}(x) \rightarrow t_n = \beta \). Consequently, \( \beta \in \mathcal{R}^*_\alpha(E') \).

So, one can deduce that

\[
\mathcal{R}^*\{\{\alpha\}\} \subseteq \mathcal{R}^*_\alpha\{\{G(\alpha)\}\}.
\] (18)

\(- \mathcal{R}^*_\alpha(E') \subseteq \mathcal{R}^*(E)\): Let us study \( \mathcal{R}^*_\alpha\{\{G(\alpha)\}\} \).

- Suppose that one rule of the form \( G(x) \rightarrow G(F_{t_i \rightarrow r_i}(x)) \) is applied. So, \( G(\alpha) \rightarrow \mathcal{G}(x) \rightarrow G(\mathcal{F}_{t_i \rightarrow r_i}(\alpha)) \). Note that \( G(x) \rightarrow \mathcal{G}(\mathcal{F}_{t_i \rightarrow r_i}(x)) \) cannot be applied as long as the symbol \( F_{t_i \rightarrow r_i} \) occurs. If \( \mathcal{R}^*_i \{F_{t_i \rightarrow r_i}(\alpha)\} = \emptyset \) then no term of \( \mathcal{T}(\mathcal{F}_{bin}) \) can be reached from \( F_{t_i \rightarrow r_i}(\alpha) \). Thus, since \( \mathcal{R}_{t_i \rightarrow r_i} \) follows the definition of \( \mathcal{R}_\alpha \) in Proposition 1, one can deduce that the rule \( t_i \rightarrow r \in \mathcal{R} \) cannot be applied on \( \alpha \).

If \( \mathcal{R}^*_i \{F_{t_i \rightarrow r_i}(\alpha)\} \neq \emptyset \) then there exists one term of \( t_1 \in \mathcal{T}(\mathcal{F}_{bin}) \) that can be reached from \( F_{t_i \rightarrow r_i}(\alpha) \). Since \( F_{t_i \rightarrow r_i}(\alpha) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_1) \), one has \( G(\alpha) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_1) \). Moreover, according to Proposition 1, \( \alpha \rightarrow_{\{t_i \rightarrow r_i\}} t_1 \). Note that \( t_1 \in \mathcal{R}^*_\alpha(E') \) since the rule \( G(x) \rightarrow x \) can be applied on \( G(t_1) \). Thus, this process can be iterated and on can build a sequence such that \( G(\alpha) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_1) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_2) \rightarrow \mathcal{R}_\alpha \mathcal{G}(t_n) \) with \( t_i \in \mathcal{R}^*_\alpha(E') \) if the rule \( G(x) \rightarrow x \) is applied on each term \( G(t_i) \) (this is possible since \( t_i \in \mathcal{T}(\mathcal{F}_{bin}) \)). Moreover, according to Proposition 1, one has \( t_i \rightarrow_{\{t_i \rightarrow r_i\}} t_{i+1} \) for \( i = 1, \ldots, n-1 \).

So one can deduce that \( t_n \in \mathcal{R}^*\{\{\alpha\}\} \). Finally, one can deduce that

\[
\mathcal{R}^*\{\{\alpha\}\} \supseteq \mathcal{R}^*_\alpha\{\{G(\alpha)\}\}.
\] (19)

- Suppose that no rule of the form \( G(x) \rightarrow G(F_{t_i \rightarrow r_i}(x)) \) is applied. Consequently, no rule of \( \bigcup_{t_i \rightarrow r_i \in \mathcal{R}_{t_i \rightarrow r_i}} \) can be applied. The unique reachable term of \( \mathcal{T}(\mathcal{F}_{bin}) \) is then obtained by applying the rules \( G(x) \rightarrow x \). So, \( \alpha \in \mathcal{R}^*_\alpha\{\{G(\alpha)\}\} \). By definition, \( \alpha \in \mathcal{R}^*\{\{\alpha\}\} \).

To conclude, for any \( \alpha \in E \), from (18) and (19), one obtains that \( \mathcal{R}^*(E) = \mathcal{R}^*_\alpha(E') \).
5 Elementary Functional Term Rewriting Systems

In this section, given a TRS $R$ over $T(F_{bin}, X) \times T(F_{bin}, X)$ and a term $t \in T(F_{bin})$, we show that it is possible to compute exactly the same set of reachable terms using a FTRS of a particular form i.e. elementary functional TRS.

**Definition 2.** Let $R_{\lambda_e}$ be a FTRS. $R_{\lambda_e}$ is said to be elementary (also called EFTRS) iff each rule of $R_{\lambda_e}$ is of one of the following forms:

1. $F(a(x, y)) \rightarrow \alpha$ with $a \in F_{bin}$, $F \in F_{NT}$, $x, y \in X$, $x \neq y$, $\alpha \in T(F_{bin} \cup F_X, X)$ and $\text{Var}(\alpha) = \{x, y\}$
2. $F(\bot) \rightarrow \alpha$ with $\alpha \in T(F_{bin} \cup F_X)$
3. $F(a(x, \bot)) \rightarrow x$ with $x \in X$, $F \in F_X$ and $a \in F_{bin}$.

Actually, EFTRSs, presented in Definition 2, are expressive enough for the specification of the whole rewriting mechanism for a rule $l \rightarrow r \in R$: 1. to find a position for rewriting, 2. to check if $l$ matches with the current subterm, then 3. to compute the resulting substitution $\sigma$, and 4. to replace the subterm by $r\sigma$.

Consequently, one has the following result.

**Proposition 2.** Let $R$ be a TRS over $T(F_{bin}, X) \times T(F_{bin}, X)$, $\alpha \in T(F_{bin})$, $\beta \in T(F_{bin})$ and $l \rightarrow r \in R$. There exists an EFTRS $R_{\lambda_e}$ such that

$$\alpha \rightarrow^* \{l \rightarrow r\} \beta \iff F_{l \rightarrow r}(\alpha) \rightarrow^*_{R_{\lambda_e}} \beta.$$

**Proof (Sketched proof).**

The key point is to construct an EFTRS $R_{\lambda_e}$ implementing the whole rewriting process. The EFTRS $R_{\lambda_e}$ is build as follows: $R_{\lambda_e} = R_{\text{visit}}^{l \rightarrow r} \cup R_{\text{TV}}^{l \rightarrow r} \cup R_{\text{check}}^{l \rightarrow r} \cup R_{\sigma}^{l \rightarrow r} \cup R_{GS}^{l \rightarrow r} \cup R_{\text{apply}}^{l \rightarrow r}$

where

- $R_{\text{visit}}^{l \rightarrow r} = \{F_{l \rightarrow r}(a(x, y)) \rightarrow a(F_{l \rightarrow r}(x), y), F_{l \rightarrow r}(a(x, y)) \rightarrow a(x, F_{l \rightarrow r}(y))\}$
- $R_{\text{TV}}^{l \rightarrow r} = \{F_{l \rightarrow r}(x) \rightarrow F_{l \rightarrow r}(\text{Check}_{l \rightarrow r}(F_{l \rightarrow r}(x)))\}$
- $R_{\text{check}}^{l \rightarrow r}$ is an EFTRS allowing us to check if a term of $T(F_{bin})$ matches with $l$. The application of this EFTRS is fired by the presence of the non-terminal symbol $F_{l \rightarrow r}$. Given a term $t \in T(F_{bin})$ matching with $l$, $(R_{\text{check}}^{l \rightarrow r})^*(\{F_{l \rightarrow r}(t)\})$ is $\{t'\}$ where
  - for all $p \in \text{Pos}_{X}(l)$, $t(p) = t'(p)$;
  - for all $p \in \text{Pos}_{X}(l)$, $t'(p) = \oplus_{x,p} \in F_{bin}$;
  - for all $p' \in \text{Pos}(t(p))$, $t(p,p') = t'(p,1.p')$.

The symbol $\oplus_{x,p}$ is a marker meaning that the term under this marker is the value that the variable $x$ at position $p$ in $l$ takes when matching occurs. When the matching between $t$ and $l$ is not possible, $(R_{\text{check}}^{l \rightarrow r})^*(\{F_{l \rightarrow r}(t)\}) = \emptyset$. Note that, at this point, we do not check the case that two identical variables have to share the same value. So $(R_{\text{check}}^{l \rightarrow r})^*(\{F_{l \rightarrow r}(t)\}) = \emptyset$ iff $t$ and $l$ are different structurally speaking.
\( R_{v \rightarrow} \) is an EFTRS allowing us to construct an ordered list of terms indexed by variables of \( l \). This list is represented by a term and semantically speaking, one can see this list as the substitution resulting from the matching step. Let \( t' \) be the term resulting from \((R_{v \rightarrow})^*([F_{\rightarrow v}(t)])\). \((R_{v \rightarrow})^*([F_{\rightarrow copy}(t')])\) leads to a unique term \( t'' \) representing the substitution resulting from the matching between \( t \) and \( l \).

\( R_{GS} \) is an EFTRS specifying the verification of a well-formed substitution. Indeed, if \( l \) is not linear then a variable \( x \) appears two times at least. If all occurrences of \( x \) share the same term as value then the substitution is well-formed. \((R_{v \rightarrow})^*(Check_{v \rightarrow}((t''))) = \{t''\} \) is the substitution is well-formed, \( \emptyset \) otherwise. \( R_{v \rightarrow} \) handles each variable occurring more than once as mentioned above.

\( R_{\sigma \rightarrow apply} \) is an EFTRS specifying the application of the well-formed substitution resulting from \( t \) on the term \( r \). Thus, \((R_{\sigma \rightarrow apply})^*([F_{\rightarrow apply}(t'')]) = \{r\sigma\} \).

The EFTRSs listed above are fully defined in Appendix.

\(- \alpha \rightarrow_{\{\rightarrow\}} \beta \Rightarrow F_{\rightarrow}\alpha \rightarrow_{\alpha, v} \beta \): Trivially, if \( \alpha \rightarrow_{\{\rightarrow\}} \beta \) then there exist a position \( p \) of \( \alpha \) and a substitution \( \sigma \) : \( X \rightarrow T(\mathbin{F}_{\binom{\text{bin}}{\text{check}}}) \) such that \( \alpha|_p = \sigma \) and \( \beta = \alpha[\sigma|_p] \). From the definition of \( R_{\alpha, v} \), one can deduce that \( F_{\rightarrow}\alpha \rightarrow_{\alpha, v} \alpha[\sigma|_p]|_p \). Thus, applying the rule \( F_{\rightarrow}\alpha \rightarrow_{\alpha, v} \alpha[\sigma|_p]|_p \). Since \( \sigma \) is a substitution from \( X \) to \( T(\mathbin{F}_{\binom{\text{bin}}{\text{check}}}) \), and according to the set of rules \( R_{\alpha, v} \cup R_{\rightarrow v} \cup R_{\rightarrow \alpha, v} \cup R_{\alpha, v, \sigma} \cup R_{\alpha, v, \sigma, \check{\alpha}, v} \), \( \alpha[\sigma|_p]|_p \) can be rewritten in \( \alpha[\sigma|_p]|_p \).

Thus, the term \( t \) represents as a term the substitution \( \sigma \). Finally, the EFTRS \( R_{\sigma \rightarrow apply} \) allows us to reconstruct \( \rho \) and the value of variables are copied symbol by symbol. The unique reachable terms is then \( \mathbin{T}(\mathbin{F}_{\binom{\text{bin}}{\text{check}}}) \). So, to summarize, \( F_{\rightarrow\check{\alpha}}(t) \rightarrow_{\alpha, v, \check{\alpha}, \check{\sigma}} \rho \sigma \in \mathbin{T}(\mathbin{F}_{\binom{\text{bin}}{\text{check}}}) \). Consequently, \( \alpha[\sigma|_p]|_p \) can be rewritten in \( \alpha[\sigma|_p]|_p \).

\(- \alpha \rightarrow_{\{\rightarrow\}} \beta \Rightarrow F_{\rightarrow}\alpha \rightarrow_{\alpha, v} \beta \): The proof for this case is close to the similar case handled in the proof of Proposition 1 in the sense that rewrite steps are ordered and depending on the initial term i.e. \( F_{\rightarrow}\alpha \). Eventually, since by hypothesis \( F_{\rightarrow}\alpha \rightarrow_{\alpha, v, \check{\alpha}, \check{\sigma}} \beta \), due to the definitions of \( R_{\rightarrow v} \) and \( R_{\check{\alpha}} \), there exists a position \( p \in \mathcal{P}(\alpha) \) such that \( F_{\rightarrow}\alpha \rightarrow_{\alpha, v, \check{\alpha}, \check{\sigma}} \beta \). Due to the stack of symbols of \( \mathbin{T}(\mathbin{F}_{\binom{\text{bin}}{\text{check}}}) \) over \( \alpha|_p \), one can show that \( F_{\rightarrow}\alpha \rightarrow_{\alpha, v, \check{\alpha}, \check{\sigma}} \beta \) is not the substitution is well-formed, \( \emptyset \) otherwise. \( R_{\check{\alpha}} \) handles each variable occurring more than once as mentioned above.
The set \( F \) Specification with EFTRS and Benchmarks

one of Theorem 1.

6.1 Specification TAP using EFTRSs

Section 6.3. Finally, benchmarks are presented in Section 6.2 describes how the verification of the property \( \phi \) defined in Section 3 can be entirely encoded in EFTRS. Finally, benchmarks are presented in Section 6.3.

6 Specification with EFTRS and Benchmarks

In Section 6.1, we present how the Tree Arbiter Protocol is encoded using EFTRS and Section 6.2 describes how the verification of the property \( \phi \) defined in Section 3 can be entirely encoded in EFTRS. Finally, benchmarks are presented in Section 6.3.

6.1 Specification TAP using EFTRSs

Let \( \mathcal{F}_{\text{NT}} \) be the set of non-terminal symbols such that \( \mathcal{F}_{\text{NT}} = \{H, R, RT, TI, I, Arbiter\} \). The set \( \mathcal{F}_{\text{bin}} \) is the one defined in Section 3, i.e., \( \{i, r, t, b, \bot\} \).

The rewriting rules (1), (2) and (3) given in Section 3 are translated as follows in EFTRS rules.

\[
H(i(x, y)) \rightarrow r(x, R(y)) \tag{20}
\]

\[
H(i(x, y)) \rightarrow r(R(x), y) \tag{21}
\]

\[
R(r(x, y)) \rightarrow r(x, y) \tag{22}
\]

\[
R(\bot) \rightarrow \bot \tag{23}
\]

Let us illustrate the rewriting of the following configuration \( t_0 = H(t(i(\bot, \bot), i(\bot, \bot))) \). None of rules mentioned above can be applied. So, no term of \( T(\mathcal{F}_{\text{bin}}) \) is reachable from \( H(t(i(\bot, \bot), i(\bot, \bot))) \).
So we need rules for specifying the circulation of the symbol $H$ before applying the rules (20), \ldots, (23). These rules are given below and are defined for each symbol of $\mathcal{F}_{bin}$ except $\bot$.

\[
\begin{align*}
H(i(x, y)) &\rightarrow i(H(x), y) \\
H(i(x, y)) &\rightarrow i(x, H(y)) \\
H(r(x, y)) &\rightarrow r(H(x), y) \\
H(r(x, y)) &\rightarrow r(x, H(y)) \\
H(b(x, y)) &\rightarrow b(H(x), y) \\
H(b(x, y)) &\rightarrow b(x, H(y)) \\
H(t(x, y)) &\rightarrow t(H(x), y) \\
H(t(x, y)) &\rightarrow t(x, H(y)) \tag{24}
\end{align*}
\]

Now, let us illustrate the use of the rules (24), \ldots, (31). Let us apply the rule (24) (resp. (25)) on $t_0$. So, the term $t'_1 = t(\overline{H(i(\bot, \bot), i(\bot, \bot)))}$ (resp. $t''_1 = t(i(\bot, \bot), \overline{H(i(\bot, \bot)))}$) is obtained. By applying rule (20) and rule (23) on $t'_1$ (resp. $t''_1$), one obtains the term $t'_2 = t(r(\bot, \bot), i(\bot, \bot))$ (resp. $t''_2 = t(i(\bot, \bot), r(\bot, \bot))$).

Note that $t'_2$ and $t''_2$ do not contain symbol of $\mathcal{F}_{NT}$. So the EFTRS can not be applied anymore. Thus, we introduce the symbol $\text{Arbiter}$ whose semantics is described by rules (42) to (45). In other words, this symbol allows us to rewrite as much as needed. The rules below fill out the specification of TAP by an EFTRS.

\[
\begin{align*}
H(t(x, y)) &\rightarrow b(x, RT(y)) \\
H(t(x, y)) &\rightarrow b(\overline{RT}(x), y) \\
\overline{RT}(r(x, y)) &\rightarrow t(x, y) \\
H(b(x, y)) &\rightarrow t(x, T(t(y))) \\
H(b(x, y)) &\rightarrow t(T(t(x), y)) \\
T(I(t(x, y)) &\rightarrow i(I(x), I(y))) \\
T(I(t(x, y)) &\rightarrow r(I(x), I(y))) \\
T(I(t(x, y)) &\rightarrow r(I(x), R(y))) \\
I(i(x, y)) &\rightarrow i(x, y) \\
I(\bot) &\rightarrow \bot \\
\text{Arbiter}(t(x, y)) &\rightarrow t(x, y) \\
\text{Arbiter}(t(x, y)) &\rightarrow \text{Arbiter}(H(t(x, y))) \tag{43} \\
\text{Arbiter}(b(x, y)) &\rightarrow b(x, y) \tag{44} \\
\text{Arbiter}(b(x, y)) &\rightarrow \text{Arbiter}(H(b(x, y))) \tag{45}
\end{align*}
\]

Now, consider the starting term $t_0 = \text{Arbiter}(t(i(\bot, \bot), i(\bot, \bot)))$. From $t_0$ and applying the rules (43), (24), (20) and (23), the term $t_1 = \text{Arbiter}(t(i(\bot, \bot), r(\bot, \bot)))$ is computed. Applying the rules (32) and (34) from $t_1$, the term $t_2$ is computed with $t_2 = \text{Arbiter}(b(i(\bot, \bot), t(\bot, \bot)))$. Note that from $t_1$ (resp. $t_2$), we can also apply the rule (42) (resp. 44) and then obtain the term $t_l(i(\bot, \bot), r(\bot, \bot))$ (resp. $b(i(\bot, \bot), I(\bot, \bot)))$. The two terms mentioned above represent two configurations actually reachable.

We note $\mathcal{K}_{\lambda}^{TAP}$ the whole EFTRS defined in this section.
6.2 Invariant Verification of TAP in Functional Reachability

In Section 3, we have defined the following property $\phi$: there is always only one token circulating in the hierarchy tree. Let us consider $\neg \phi$. In words, $\neg \phi$ specifies that there may exist one configuration containing zero token or two tokens at least.

The following EFTRS $\mathcal{R}_{X^0}^{-\phi}$ specifies the property $\neg \phi$.

\[
\begin{align*}
\neg \omega(r(x, y)) & \rightarrow r(\neg \omega(x), \neg \omega(y)) \\
\neg \omega(b(x, y)) & \rightarrow b(\neg \omega(x), \neg \omega(y)) \\
\neg \omega(i(x, y)) & \rightarrow i(\neg \omega(x), \neg \omega(y)) \\
\neg \omega(t) & \rightarrow \bot \\
\text{Two}(r(x, y)) & \rightarrow r(\text{Two}(x), y) \\
\text{Two}(b(x, y)) & \rightarrow b(\text{Two}(x), y) \\
\text{Two}(i(x, y)) & \rightarrow i(\text{Two}(x), y) \\
\text{Two}(t) & \rightarrow i(\text{Two}(x), y)
\end{align*}
\]

Semantically speaking, the symbol $\neg \omega \in \mathcal{F}_{NT}$ is introduced for checking that a term does not contain the symbol $t$. Indeed, there is no rule of the form $\neg \omega(t(x, y)) \rightarrow \beta$. Consequently, for a term $\alpha \in T(\mathcal{F}_{bin})$ for which there exists $p \in \text{Pos}(\alpha)$ such that $\alpha(p) = t$, $(\mathcal{R}_{X^0}^{-\phi})^*(\{\neg \omega(\alpha)\}) = \emptyset$.

The symbol $\text{Two} \omega \in \mathcal{F}_{NT}$ is introduced for checking that the symbol $t$ occurs in a term one time at least. So, for a term $\alpha \in T(\mathcal{F}_{bin})$,

- $(\mathcal{R}_{X^0}^{-\phi})^*(\{\text{Two} \omega(\alpha)\}) = \emptyset$ means that the symbol $t$ occurs in $\alpha$ only once or never;
- $(\mathcal{R}_{X^0}^{-\phi})^*(\{\text{Two} \omega(\alpha)\}) \neq \emptyset$ means that the symbol $t$ occurs in $\alpha$ at least two time.

Consequently,

the TAP satisfies $\phi \iff (\mathcal{R}_{X^0}^{-\phi})^*((\mathcal{R}_{AP}^T)^*(\{\text{Two} \omega(Two \omega(t_0)), \neg \omega(t_0)\})) = \emptyset$

where $t'_0 = \text{Arbiter}(t_0)$ and $t_0$ is the initial configuration of the TAP.

Using our tool, we have been able to check that $\phi$ is true for the TAP handling $2^{400}$ processors in $2.933s$.

6.3 Some Experiments

Hereafter reported results have been obtained on a PC 1.7GHz 1GB RAM. We have compared our tool with Timbuk [31, 27] and Maude [19]. Timbuk is a collection of tools for achieving proofs of reachability over TRSs. This tool manipulates tree automata for representing possibly infinite set of terms. Maude is rewrite engine which is often used in a verification context. The table below summarizes
some results obtained on well-known protocols i.e. Tree Arbiter Protocol (TAP), Percolate Protocol (PP) and Leader Election Protocol (LEP). For this experiments, we have adapted local fixpoint techniques in our EFTRS framework. Of course, acceleration techniques are well adapted for this kind of protocols where the number of configurations may be huge but finite. The acceleration techniques can be implemented automatically for all of these study cases. Thus, it explains the difference in terms of computation times between tools. However, Maude and Timbuk can be used for the verification of infinite systems [23, 28, 29, 41, 5]. For now, our technique is exclusively defined for finite systems (and thus terminating TRS).

In the following table, $\Sigma$ denotes the study case, $N$ the number of processors at the bottom of the trees and Confs the size of the state space to explore. The number of configurations is evaluated with a Java program, thus, "?" denotes that the library `java.math.BigDecimal` we used cannot return the result within three hours. For further details of model descriptions and the benchmarks, see [4].

<table>
<thead>
<tr>
<th>$\Sigma$</th>
<th>$N$</th>
<th>Confs</th>
<th>TRS</th>
<th>EFTRS</th>
</tr>
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<td></td>
<td></td>
<td></td>
<td>Timbuk</td>
<td>Maude</td>
</tr>
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<td>$&gt;3h$</td>
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<td>-</td>
</tr>
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<td>-</td>
</tr>
<tr>
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<td>$2^{400}$</td>
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7 Applications of EFTRS

7.1 Application on Petri net and Experimental results


Dining philosophers problem is an illustrative example of a common computing problem in concurrency. This classic model is obtained by connecting N
identical submodels (see Figure 2), one per philosopher, in a circular fashion. Each philosopher starts in the idle state and occasionally decides to eat: to do so he must acquire both his left and right forks, which he then releases when he has finished eating.

On the first phase, construct a term for sub-net level: $\text{local}(u, v)$ where $u$ and $v$ is the encoding of the identical model, e.g. $\text{local}(\bot, \text{hasL}(\bot, \bot), \text{hasR}(\true, \bot), ...))$.

On the second phase, construct a hierarchical model by combining two sub-nets like $\text{sys}(\text{local}(u, v), \text{local}(u, v))$. From this 1-level model, we can construct 2-level by the same way, and so forth.

There is relations between two neighbours. So we have a synchronization operator by going down from root to sub-net levels:

$$H(\text{sys}(x, y)) \rightarrow \text{SyncIPP}(\text{sys}(x, y))$$
$$H(\text{sys}(x, y)) \rightarrow \text{SyncN}(\text{sys}(x, y))$$
$$\text{SyncIPP}(\text{sys}(x, y)) \rightarrow \text{sys}(\text{SyncIPP}(x), y)$$
$$\text{SyncN}(\text{sys}(x, y)) \rightarrow \text{sys}(x, \text{SyncN}(y))$$

Go down from $k$-level to $k-1$-level of model:

$$\text{SyncIPP}(\text{sys}(x, y)) \rightarrow \text{sys}(\text{SyncIPP}(x), \text{SyncIPP}(y))$$

Then at each level of the model:

$$\text{SyncIPP}(\text{sys}(x, y)) \rightarrow \text{sys}(\text{SyncIPP}(x), y)$$
$$\text{SyncN}(\text{sys}(x, y)) \rightarrow \text{sys}(\text{SyncN}(y))$$

Finally, at sub-net levels of the model:

$$\text{SyncIPP}(\text{local}(x, y)) \rightarrow \gamma(x, y)$$
$$\text{SyncN}(\text{local}(x, y)) \rightarrow \alpha(x, y)$$

In the other hand, at sub-net levels of the model:

$$\text{Local}(\text{local}(x, y)) \rightarrow \gamma(x, y)$$

where $\alpha, \beta, \gamma$ are compositions of some primitive rules $\text{Post}_p$ and $\text{Pre}_p$ for each place $p$:

$$\text{Post}_p(p(x, y)) \rightarrow p(\text{Post}(x), y)$$
$$\text{Pre}_p(p(x, y)) \rightarrow p(\text{Pre}(x), y)$$

Our implementation shows that our automatical Petri-EFTRS transformation gives the same configurations number of some Petri net model with other approaches.
The following experiments deal with Petri Net problems. We also compare our tool with some tools like SPIN [35], NuSMV [17], SMART [10, 15, 11] and HSDD [46, 47]. These values concerning the other tools are directly taken from [16, 9, 22, 46, 47]. Some missing results are indicated by "N/A".

<table>
<thead>
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<th>N</th>
<th>Conf</th>
<th>SPIN</th>
<th>SMV</th>
<th>SMART</th>
<th>HSDD</th>
<th>EFTRS</th>
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<td>N/A</td>
<td>-</td>
<td>-</td>
<td>0.01</td>
<td>2.77</td>
</tr>
<tr>
<td>2^25</td>
<td>?</td>
<td>N/A</td>
<td>-</td>
<td>-</td>
<td>0.02</td>
<td>135.4</td>
</tr>
</tbody>
</table>

**Dining Philosophers.**

<table>
<thead>
<tr>
<th>N</th>
<th>Conf</th>
<th>SPIN</th>
<th>SMV</th>
<th>SMART</th>
<th>HSDD</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^5</td>
<td>5136</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.13</td>
</tr>
<tr>
<td>2^10</td>
<td>6.80E+7</td>
<td>8.2</td>
<td>0.13</td>
<td>0.06</td>
<td>0.0</td>
<td>18.53</td>
</tr>
<tr>
<td>2^15</td>
<td>1.65E+16</td>
<td>-</td>
<td>2851</td>
<td>0.18</td>
<td>0.03</td>
<td>2398</td>
</tr>
</tbody>
</table>

**Slotted Ring Protocol.**

<table>
<thead>
<tr>
<th>N</th>
<th>Conf</th>
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<th>SMV</th>
<th>SMART</th>
<th>HSDD</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
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<td>2.359E+6</td>
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<tr>
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<td>0.2</td>
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<td>-</td>
<td>-</td>
<td>7.04</td>
<td>1.0</td>
<td>1115</td>
</tr>
</tbody>
</table>

**Round Robin Mutex Protocol [32].**

1. **Dining philosophers problem.** The hierarchical model can resolve problem having size until \(N = 2^{35}\).

<table>
<thead>
<tr>
<th>N</th>
<th>#Conf</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^5</td>
<td>1.155E+20</td>
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<tr>
<td>2^10</td>
<td>1.023E+642</td>
<td>1.251</td>
</tr>
<tr>
<td>2^15</td>
<td>2.094E+20544</td>
<td>1.966</td>
</tr>
<tr>
<td>2^20</td>
<td>1.868E+657418</td>
<td>2.779</td>
</tr>
<tr>
<td>2^25</td>
<td>?</td>
<td>3.661</td>
</tr>
<tr>
<td>2^30</td>
<td>?</td>
<td>4.782</td>
</tr>
<tr>
<td>2^35</td>
<td>?</td>
<td>135.407</td>
</tr>
</tbody>
</table>

We can also compare model size \(N\) with some related results \(^1\):

<table>
<thead>
<tr>
<th>EFTRS</th>
<th>SPIN</th>
<th>NuSMV</th>
<th>SDD</th>
<th>SMART</th>
<th>HSDD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{20}</td>
<td>-</td>
<td>200</td>
<td>5000</td>
<td>5000</td>
<td>2^{200000}</td>
</tr>
</tbody>
</table>

2. **Slotted Ring Protocol.** The hierarchical model can resolve problem having size until \(N = 2^4\).

<table>
<thead>
<tr>
<th>N</th>
<th>#Conf</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^4</td>
<td>5136</td>
<td>1.131</td>
</tr>
<tr>
<td>2^9</td>
<td>6.802E+7</td>
<td>18.537</td>
</tr>
<tr>
<td>2^14</td>
<td>1.651E+16</td>
<td>2398.754</td>
</tr>
</tbody>
</table>

\(^1\) Some are missing as indicated by "/\).
We can also compare model size $N$ with some related results:

<table>
<thead>
<tr>
<th>$N$</th>
<th>EFTRS</th>
<th>SPIN</th>
<th>NuSMV</th>
<th>SDD</th>
<th>SMART</th>
<th>HSDD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^1$</td>
<td>2</td>
<td>6</td>
<td>15</td>
<td>50</td>
<td>300</td>
<td>200</td>
</tr>
<tr>
<td>$2^2$</td>
<td>2</td>
<td>6</td>
<td>15</td>
<td>50</td>
<td>300</td>
<td>200</td>
</tr>
</tbody>
</table>

3. **Round Robin Mutex Protocol** (presented in [32]).

This protocol solves a specific type of mutual exclusion problem among $N$ processes organized in a circular fashion, requiring access to a shared resource.

The hierarchical model can resolve problem having size until $N = 2^8$. We can also compare model size $N$ with some related results$^2$:

<table>
<thead>
<tr>
<th>$N$</th>
<th>EFTRS</th>
<th>SPIN</th>
<th>NuSMV</th>
<th>SDD</th>
<th>SMART</th>
<th>HSDD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>16</td>
<td>60</td>
<td>-</td>
<td>1100</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>

Saturation approach (presented by Ciardo et al. in [13] for MDD, a BDD-like structure). This technique is also called local fixed point (LFP) technique which fights the intermediate peak size effect in the structure like d-BDD as DDD [21], SDD [22], HSDD [46, 45].

### 7.2 Application of EFTRS on LTL model-checking

[48, 20] show us how transform a LTL formula to Büchi Automata. The problem now is to encode a Büchi automaton into FTRS (or EFTRS) like $\text{buchi}(u, v)$ then construct the synchronized product of Büchi automaton and the system is $\text{ltl}(\text{buchi}(u, v), \text{sys}(w, t))$ where $\text{sys}(w, t)$ is an encoding of system and $u, v, w, t \in \mathcal{F}_{bin}$.

Operations on the synchronized product is defined for each property $\varphi$ as the following: $\text{Sync}_\varphi(\text{ltl}(\text{buchi}(u, v), \text{sys}(w, t))) \rightarrow \text{ltl}(\text{Buchi}_\varphi(\text{buchi}(u, v)), \text{Sys}_\varphi(\text{sys}(w, t)))$ where

- $\text{Buchi}_\varphi$ is $\text{Change}, \varphi_{s,x}(s(x, y)) \rightarrow s'(x, y)$
- $\text{Sys}_\varphi$ is a composition of some operations on the system: $\text{Succ}$ compute the next system states and $\text{Test}_\varphi$ works as a filter keeping only states satisfying $\varphi$.

$^2$ These values were not measured by us, but are directly taken from [16, 22, 9].
In Dining philosophers problem, since forks can be acquired in any order, this model is known to have two deadlock states, the one where each philosopher has acquired his left fork and waits for his right fork (which, being also the left fork of his right neighbor, will never be released), and the symmetric state where each philosopher has acquired his right fork.

Thus, we are interested in the behavior such as: In a moment in the future, the first philosopher has the left but he never has the right by using the LTL formula $F_1 = F(p_1.\text{hasL} \land G(\neg p_1.\text{hasR}))$ (See Büchi automaton in Figure 3a).

We are also interested in the behavior such as: First the first philosopher has the right then put this fork and finally his neighbor has the left by using the LTL formula $F_2 = (p_1.\text{hasR} \cup (p_1.\text{Fork} \cup p_N.\text{hasL}))$. See this Büchi automaton in Figure 3b.

An first implementation of LTL model-checker based on EFTRS of these LTL formulas is in the below table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$#\text{Conf}_{F_1}$</th>
<th>$#\text{Conf}_{F_2}$</th>
<th>$F_1$</th>
<th>$F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>28657</td>
<td>3.139</td>
<td>1.036E+5</td>
<td>1.777</td>
</tr>
<tr>
<td>4</td>
<td>2.971E+9</td>
<td>46.584</td>
<td>1.074E+10</td>
<td>8.936</td>
</tr>
<tr>
<td>5</td>
<td>3.194E+19</td>
<td>5486.060</td>
<td>1.155E+20</td>
<td>251.871</td>
</tr>
</tbody>
</table>

Fig. 3: Büchi automatons a) $F_1$ and b) $F_2$

8 Conclusion

In this paper, we have introduced a new formalism, so called FTRS, for the specification of transition relation for finite system verification. We have shown that the expressiveness of FTRSs and EFTRSs is the same as classical FTRSs. By adapting acceleration techniques for our new formalism and for the protocol to study, the computation times are really convincing. Moreover, the exploration of huge state spaces has been possible with our tool when some others fail. Some experiments have been carried out in the field of Petri nets. Actually, our results are not so good as expected, but we plan to refine the transformation Petri nets towards EFTRS [4] in a couple of months.

A first step towards the verification of invariants on finite systems has been initiated also in this paper. Some investigations are carried out on the field of the LTL model-checking. And it would look like possible to express the whole LTL model-checking with FTRS [4].
References


http://eftrs.svn.sourceforge.net/viewvc/eftrs/.


20 Yohan Boichut, Jean-Michel Couvreur, Duy-Tung Nguyen


http://nusmv.irst.itc.it/.


A EFTRS

Definition 3 ($\mathcal{R}_{\text{check}}^{\rightarrow}$). Let $l \rightarrow r$ be a rule of $\mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X})$. We denote by $\mathcal{R}_{\text{check}}^{\rightarrow}$ the elementary FTRS built from $l$ as follows:

1. $F_{1}^{p}(x) \rightarrow \bigoplus_{i \in p}(x) \in \mathcal{R}_{\text{check}}^{\rightarrow}$ if $p \in \text{Pos}_{\mathcal{X}}(l)$ and where $\bigoplus_{i \in p}$ is a special symbol that we consider to be in $\mathcal{F}_{\text{bin}}$
2. $F_{1}^{p}(a(x,y)) \rightarrow a(F_{1}^{p}(x), F_{1}^{p}(y)) \in \mathcal{R}_{\text{check}}^{\rightarrow}$ if $p \in \text{Pos}(l) \setminus \mathcal{F} \text{Pos}(l)$ and $a = l(p)$
3. $F_{1}^{p}(\bot) \rightarrow \bot \in \mathcal{R}_{\text{check}}^{\rightarrow}$ if $p \in \mathcal{F} \text{Pos} \setminus \text{Pos}_{\mathcal{X}}(l)$

Example 1. Consider the term $a(b(x, \bot), c(d(\bot, y), z)) \rightarrow r$ be a rewrite rule, $x, y, z \in \mathcal{X}$ and $\text{Var}(r) \subseteq \{x, y, z\}$. For a matter of readability, we denote this term by $l$. Thus,

$$
\mathcal{R}_{\text{check}}^{\rightarrow} = \left\{ 
\begin{array}{l}
F_{1}^{c}(a(x, y)) \rightarrow a(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(b(x, y)) \rightarrow a(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow a(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow c(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow c(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow c(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow c(F_{1}^{c}(x), F_{1}^{c}(y)) \\
F_{1}^{c}(c(x)) \rightarrow c(F_{1}^{c}(x), F_{1}^{c}(y))
\end{array}
\right\}
$$

We now introduce the notion of variable tree built from a term.

Definition 4. Let $t$ be a term of $\mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X})$. Let $x_{0}, \ldots, x_{n}$ be the variables of $t$ occurring in $l$ from the leftmost variable of $t$ to the rightmost variable of $t$. The empty variable tree (variable tree for short) of $t$, denoted by $\top_{\mathcal{X}}(t)$, is defined by the following term: $\top_{\mathcal{X}}(x_{0}(\bot), \top_{\mathcal{X}}(x_{1}(\bot), \ldots, \top_{\mathcal{X}}(x_{n}(\bot), \bot_{\mathcal{X}}) \ldots))$.

Definition 5 ($\mathcal{R}_{\sigma}^{\rightarrow \text{compute}}$). Let $l \rightarrow r$ be a rule of $\mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}_{\text{bin}}, \mathcal{X})$. Let $x_{0}, \ldots, x_{n}$ be the set of variables occurring in $l$ and that can be read from the left to the right in $l$. Thus, the EFTRS $\mathcal{R}_{\sigma}^{\rightarrow \text{compute}}$ is defined as follows:

1. $F_{1}^{\text{opp}}(x) \rightarrow D_{\text{opp}}(RW_{\text{opp}}(RW_{x_{1}}(\ldots RW_{x_{n}}(\top(F_{1}^{x_{1}}(x), \top_{\mathcal{X}}(t)))) \ldots))$
2. $RW_{x_{1}}(x) \rightarrow RW_{x_{1}}(RW_{x_{i}}(x))$
3. $NF_{X_{1}}(x) \rightarrow NF_{X_{1}}(x)$
4. $\{ NF_{X_{1}}(a(x, y)) \rightarrow a(NF_{X_{1}}(x), NF_{X_{1}}(y)) | a \in \mathcal{F}_{\text{bin}} \}$
5. $\{ NF_{X_{1}}(\bigoplus_{i \in x}(x)) \rightarrow \bigoplus_{i \in x}(x) | x_{i} \neq x_{i} \}$
6. $NF_{X_{1}}(\bot_{\mathcal{X}}) \rightarrow \bot_{\mathcal{X}}$
7. $NF_{X_{1}}(\top_{\mathcal{X}}) \rightarrow \top_{\mathcal{X}}$
8. $NF_{X_{1}}(\bot_{\mathcal{X}}) \rightarrow \bot_{\mathcal{X}}$
9. $RW_{x_{1}}(x) \rightarrow RW_{x_{1}}(x) \text{ with } a \in \mathcal{F}_{\text{bin}}$
10. $RW_{x_{1}}(\top(x, y)) \rightarrow RW_{x_{1}}(\top(x, y)) \text{ with } a \in \mathcal{F}_{\text{bin}}$
11. $\{ SW_{x_{1}}^{0}(\top(x, y)) \rightarrow SW_{x_{1}}^{0}(\top(x, y)) \text{ with } a \in \mathcal{F}_{\text{bin}}$
Definition 6 \((R_{GS}^l)\). Let \(l \rightarrow r\) be a rule of \(T(F_{bin}, \mathcal{X}) \times T(F_{bin}, \mathcal{X})\). Let \(x_0, \ldots, x_n\) be the variables from the leftmost one of \(l\) to the rightmost one. Let Indexes be the set of pairs of variable indexes corresponding to the same initial variable. Let \(R_{GS}^l\) be the EFTRS dedicated to check that two variables have the same term on the instantiated variable tree. Thus, \(R_{GS}^l\) is composed of the following rules:

1. \(Check_{\rightarrow l}(x) \rightarrow CFor_{i,j}(CFor_{i-1,j-1}(\ldots CFor_{i,j}(x) \ldots))\) with Indexes = \(\{(i_1, j_1), \ldots, (i_n, j_n)\}\)
2. \(PutMarks_{i,j}(\top(x,y)) \rightarrow \top(x, PutMarks_{i,j}(y))\) for \(i > 0\)
3. \(PutMarks_{i,j}(\top(x,y)) \rightarrow \top(\overline{x}, y)\)
4. \(CFor_{i,j}(x) \rightarrow Test_{\rightarrow l}(PutMarks_{i,j}(x))\)
5. \(Test_{\rightarrow l}(x) \rightarrow Test_{\rightarrow l}(R_{GS}^l(x))\)
6. \(Test_{\rightarrow l}(x) \rightarrow \top_{x,p}(x)\)
7. \(\{R_{GS}^l(x) \rightarrow R_{GS}^l(y)\}\) with \(y \in F_{bin}\)
8. \(\{R_{GS}^l(x) \rightarrow \top(x, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
9. \(\{R_{GS}^l(x) \rightarrow \top(\overline{x}, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
10. \(\{R_{GS}^l(x) \rightarrow \top(x, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
11. \(\{R_{GS}^l(x) \rightarrow \top(\overline{x}, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
12. \(\{R_{GS}^l(x) \rightarrow \top(x, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
13. \(\{R_{GS}^l(x) \rightarrow \top(\overline{x}, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
14. \(\{R_{GS}^l(x) \rightarrow \top(x, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)
15. \(\{R_{GS}^l(x) \rightarrow \top(\overline{x}, R_{GS}^l(y))\}\) with \(y \in F_{bin}\) and \(i > 0\)

Definition 7 (Substitution of variables by markers). Let \(l \rightarrow r\) be a rewrite rule of \(T(F_{bin}, \mathcal{X}) \times T(F_{bin}, \mathcal{X})\). We denote by \(\sigma_{l \rightarrow r}\) a marker substitution built such that:

\[\sigma_{l \rightarrow r} = \{x \mapsto \overline{x}(\bot) \mid x \in \text{Var}(r)\}.\]

Definition 8 \((R_{\sigma_{l \rightarrow r}}^{l \rightarrow r})\). Let \(l \rightarrow r\) be a rule of \(T(F_{bin}, \mathcal{X}) \times T(F_{bin}, \mathcal{X})\). Thus, \(R_{\sigma_{l \rightarrow r}}^{l \rightarrow r}\) contains the following rules:

1. \(F_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x) \rightarrow D_{\sigma_{l \rightarrow r}}(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(\ldots RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(\top(\sigma_{l \rightarrow r}, x) \ldots)))\)
2. \(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x) \rightarrow RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x))\)
3. \(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x) \rightarrow NMA_{\sigma_{l \rightarrow r}}(x)\)
4. \(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x) \rightarrow RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(x)\) with \(a \in F_{bin}\)
5. \(RW_{\sigma_{l \rightarrow r}}^{l \rightarrow r}(\top(x,y)) \rightarrow \top(W_{\sigma_{l \rightarrow r}}(x), SR_{\sigma_{l \rightarrow r}}(M^{\sigma_{l \rightarrow r}}(\top(x,y))))\) with \(a \in F_{bin}\)
6. \(M^{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow \top(\overline{\sigma_{l \rightarrow r}(x,y)}) \cup (M^{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow \top(x, M^{\sigma_{l \rightarrow r}}(\top(x,y))) \) with \(i > 0 \wedge i \leq n \wedge x \in \text{Var}(r)\)
7. \(\{SR_{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow \top(R_{\sigma_{l \rightarrow r}}(x,y))\} \cup \{SR_{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow \top(x, SR_{\sigma_{l \rightarrow r}}(x,y))\} \) with \(i > 0 \wedge i \leq n\)
8. \(D_{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow D_{\sigma_{l \rightarrow r}}(\top(x,D(y)))\)
9. \(D_{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow x\)
10. \(D_{\sigma_{l \rightarrow r}}(\top(x,y)) \rightarrow D'(y)\)
11. \(D'(\top(x,y)) \rightarrow x\)
12. \(D'(\top(x,y)) \rightarrow x\)
Fig. 4 illustrates the copy of the value stored in the variable $x_2$. The process is the same for the variables $x_1$ and $x_0$. 
Fig. 4: Copy of the value stored in $x_2$
B Model-checking of TRS, FTRS and EFTRS

B.1 Simple models

**Tree Arbiter Protocol** In paper submitted to FMCAD2010.

**Percolate Protocol** Program percolate consists of a tree of processes, each of them has its local label, which ranges over the set of values "p", "n", "u". The values "p" (or "n") should be interpreted as "positif" (or "negatif"), the value "u" should be interpreted as "undefined yet", which implies that it will eventually change to either "p" or "n".

Initially, all the leaf processes in the tree are presented as "p"(⊥, ⊥) or "n"(⊥, ⊥), and all other processes have value "u". The purpose of this model is to percolate to the root of the tree a value "p" if at least one of the leaves has value "p", and a value of "n", if all leaves have value "n". If a process does not yet have a defined value but all its childrens' values are defined then the process sets its value to the disjunction of the values of its children.

Consequently, we can represent a configuration of program percolate as a tree over the alphabet \{"p", "n", "u", ⊥\} where "p", "n", "u" are binary symbols and ⊥ is a special nullary symbol (see the figure 5).

**TRSs** \(R_1\):

\[
\begin{align*}
    u(n(x, y), n(z, t)) & \rightarrow n(n(x, y), n(z, t)) \\
    u(p(x, y), z) & \rightarrow p(p(x, y), z) \\
    u(x, p(y, z)) & \rightarrow p(x, p(y, z))
\end{align*}
\]

**Corresponding FTRSs:**

- **Generated rules (Rules with the left side \(H(u(\ldots))\))**
  \[
  \begin{align*}
  H(u(n(x, y), n(z, t))) & \rightarrow n(n(x, y), n(z, t)) \\
  H(u(p(x, y), z)) & \rightarrow p(p(x, y), z) \\
  H(u(x, p(y, z))) & \rightarrow p(x, p(y, z))
  \end{align*}
  \]

- **Circulation (Rules with the left side \(H(p(x, y)), H(n(x, y))\) are not necessary generated here)**
  \[
  \begin{align*}
  H(u(x, y)) & \rightarrow u(H(x), y) \\
  H(u(x, y)) & \rightarrow u(x, H(y))
  \end{align*}
  \]

- **Fixed point**
  \[
  \begin{align*}
  \text{Percolate}(x) & \rightarrow x \\
  \text{Percolate}(x) & \rightarrow \text{Percolate}(H(x))
  \end{align*}
  \]

**Corresponding EFTRS:**

- **Generated rules**
  \[
  \begin{align*}
  H(u(x, y)) & \rightarrow n(\text{Neg}(x), \text{Neg}(y)) \\
  H(u(x, y)) & \rightarrow p(\text{Pos}(x), y) \\
  H(u(x, y)) & \rightarrow p(x, \text{Pos}(y)) \\
  \text{Neg}(n(x, y)) & \rightarrow n(x, y) \\
  \text{Pos}(p(x, y)) & \rightarrow p(x, y)
  \end{align*}
  \]

- **Circulation**
  \[
  \begin{align*}
  H(u(x, y)) & \rightarrow u(H(x), y) \\
  H(u(x, y)) & \rightarrow u(x, H(y))
  \end{align*}
  \]

- **Fixed point**
  \[
  \begin{align*}
  \text{Percolate}(u(x, y)) & \rightarrow u(x, y) \\
  \text{Percolate}(u(x, y)) & \rightarrow \text{Percolate}(H(u(x, y)))
  \end{align*}
  \]
Leader Election Protocol A set of processes, denoted by the leaves, want to elect a leader. Each of them decides first whether to be a candidate or not. The election process proceeds in two phases.

The first phase consists of the internal nodes polling their children nodes to see if at least one of them is candidate. In such a case, the internal node becomes a candidate as well and will be labelled "p", otherwise it will be labelled "n".

Similarly as the program Percolate, all the leaf processes in the tree have value "n" and all other processes have value "p". The purpose of the first phase is to percolate to the root of the tree a value if at least one of the leaves has value "p", and a value of "n", if all leaves have value "n." If a process does not yet have a defined value but all its childrens’ values are defined then the process sets its value to the disjunction of the values of its children.

The second phase is the actual election procedure. In case the root assigned "n", we have no candidate and can not continue the election. From the result of the first phase, we prepare for the second phase by marking root by a label "c" (see the figure 7). The root chooses (elects) one candidate non-deterministically among its children labelled "p". An internal node that has been elected (noted by the label "c"), elects in turn one of its children that declared itself candidate.

TRSs $R_1$ and corresponding functional TRSs can give the same result set $E_1$ from an initial set $E_0$: $E_1 = R_1^*(E_0) = Percolate(E_0)$.

We are interested only in a configuration of $E_1$ or $R_1^*(E_0)$ that contains only labels "n", "p" and $\perp$ (see the figure 6). We can find the final result $E_1'$ by using $Inv(Percolate(E_0))$. Further detail of FTRS about $Inv$ in Section B.2.

Corresponding FTRS:

- Generated rules
  
  \[
  H(c(p(x, y), p(z, t))) \rightarrow c(c(x, y), p(z, t))
  
  H(c(p(x, y), p(z, t))) \rightarrow c(p(x, y), c(z, t))
  
  H(c(p(x, y), n(z, t))) \rightarrow c(c(x, y), n(z, t))
  
  H(c(n(x, y), p(z, t))) \rightarrow c(n(x, y), c(z, t))
  \]

- Circulation (Rules with the left side $H(p(z, y))$, $H(n(z, y))$ are not necessary generated here)
  
  \[
  H(c(x, y)) \rightarrow c(H(x), y)
  
  H(c(x, y)) \rightarrow c(x, H(y))
  \]

- Fixed point
  
  Election(x) $\rightarrow$ z
  
  Election(x) $\rightarrow$ Election(H(x))

Corresponding EFTRS:

- Generated rules
  
  \[
  H(c(x, y)) \rightarrow c(Elec(x), Neg(y))
  
  H(c(x, y)) \rightarrow c(Elec(x), Pos(y))
  \]
Invariants are the most common and useful of safety properties, that is, properties stating that some bad things should never happen. Given a system and an initial state \( \text{init} \), an invariant \( \text{Inv} \) is a predicate defining a subset of states having two properties:

- it contains \( \text{init} \),
- it contains all of reachable states from \( \text{init} \).

We check if the set of bad states \( \text{Bad} = \neg \text{Inv} = \mathcal{R}^*(\text{init}) \setminus \text{Inv}(\mathcal{R}^*(\text{init})) \) is empty.

For example, given a term \( \text{clock} = \text{time}(i(\bot, \bot), \bot) \), and a FTRS: \( \mathcal{R} = \{ H(\text{time}(x, y)) \rightarrow \text{time}(H(x), y), H(0(x, y)) \rightarrow 1(x, y), H(1(x, y)) \rightarrow 2(x, y), \ldots, H(23(x, y)) \rightarrow 0(x, y) \} \) with \( i = 1..23 \in \mathcal{F}_{bin} \).

Here we want to verify if \( \text{time}(i(\bot, \bot), \bot) \) has \( i = 1..23 \in \mathcal{F}_{bin} \) by launching a test \( \mathcal{R}_{\text{test}} = \{ \text{Inv}(\text{time}(x, y)) \rightarrow \text{time}(\text{Inv}(x), y), \text{Inv}(0(x, y)) \rightarrow 0(x, y), \text{Inv}(1(x, y)) \rightarrow 1(x, y), \ldots, \text{Inv}(23(x, y)) \rightarrow 23(x, y) \} \).

1. **Tree Arbiter Protocol.** In paper submitted to FMCAD2010.

   We have another solution: Compute \( \text{Inv} = \text{Inv}_{\text{OnlyT}}(\mathcal{R}^*(\text{init})) \):

   \[
   \begin{align*}
   \text{Inv}_{\text{OnlyT}}(\text{init}) & \rightarrow a(\text{Inv}_{\text{OnlyT}}(\text{init}), \text{Inv}_{\text{OnlyT}}(\text{init})) \\
   \text{Inv}_{\text{OnlyT}}(t(x, y)) & \rightarrow t(\text{Inv}_{\text{OnlyT}}(x), \text{Inv}_{\text{OnlyT}}(y)) \\
   \text{Inv}_{\text{OnlyT}}(a(x, y)) & \rightarrow a(\text{Inv}_{\text{OnlyT}}(a), \text{Inv}_{\text{OnlyT}}(a)) \\
   \text{Inv}_{\text{OnlyT}}(\bot) & \rightarrow \bot
   \end{align*}
   \]
with $a \in \{i, r, b\}$

We also show how we can find this results in TRS: We must use markers that always provide an explosion of system states.

Let begin from the bottom and go upward step by step, we will replace each node by "conflict" if all of his children are "onlyt" or at least one is "conflict", by "not" if all of his children are "not", and by "onlyt" in otherwise.

TRS:

\[
\begin{align*}
\text{a}(\text{conflict}(x, y), z) & \rightarrow \text{conflict}(\text{conflict}(x, y), z) \\
\text{a}(x, \text{conflict}(y, z)) & \rightarrow \text{conflict}(x, \text{conflict}(y, z)) \\
\end{align*}
\]

\[
\begin{align*}
\text{a}(\text{onlyt}(x, y), \text{onlyt}(z, w)) & \rightarrow \text{onlyt}(\text{onlyt}(x, y), \text{onlyt}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{conflict}(\text{onlyt}(x, y), \text{not}(z, w)) \\
\text{a}(\text{not}(x, y), \text{not}(z, w)) & \rightarrow \text{not}(\text{not}(x, y), \text{not}(z, w)) \\
\text{a}(\text{not}(x, y), \text{onlyt}(z, w)) & \rightarrow \text{not}(\text{not}(x, y), \text{onlyt}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{not}(\text{not}(x, y), \text{not}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{onlyt}(\text{onlyt}(x, y), \text{not}(z, w)) \\
\text{a}(\text{not}(x, y), \text{onlyt}(z, w)) & \rightarrow \text{onlyt}(\text{not}(x, y), \text{onlyt}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{onlyt}(\text{onlyt}(x, y), \text{not}(z, w)) \\
\text{a}(\text{not}(x, y), \text{onlyt}(z, w)) & \rightarrow \text{onlyt}(\text{not}(x, y), \text{not}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{conflict}(\text{onlyt}(x, y), \text{not}(z, w)) \\
\text{a}(\text{not}(x, y), \text{onlyt}(z, w)) & \rightarrow \text{conflict}(\text{not}(x, y), \text{onlyt}(z, w)) \\
\text{a}(\text{onlyt}(x, y), \text{not}(z, w)) & \rightarrow \text{conflict}(\text{onlyt}(x, y), \text{not}(z, w)) \\
\end{align*}
\]

2. **Percolate Protocol.** An initial set $E_0$ can be depicted in the figure 5.

![Figure 5: Initial set $E_0$](image_url)

We are interested in tree containing only labels "n", "p" and $\perp$ (see the figure 6):

\[
E_1 = (R_1'(E_0) \cap T(\{n, p, \perp\})) = \text{Inv(Percolate}(E_0)).
\]

Corresponding FTRSs is described below: $\text{Inv}(\perp) \rightarrow \perp$

$\text{Inv}(p(x, y)) \rightarrow p(\text{Inv}(x), \text{Inv}(y))$
$Inv(n(x, y)) \rightarrow n(Inv(x), Inv(y))$

Finally, we consider the sub tree whose root is labeled by "p":

$$E'_1 = (E_1 \cap \{p(t_1, t_2) | t_1, t_2 \in T\}) = P(E_1).$$

where $P(p(x, y)) \rightarrow p(x, y)$.

For conclusion, $E'_1 \neq \emptyset$ means that at least one of the leaves has value "p", $E'_1 = \emptyset$ means that all leaves have value "n".

![Fig. 6: $E'_1 = Inv(Percolate(E_0))$](image)

3. **Leader Election Protocol.** From the result of the first phase (see the figure 6), we prepare for the second phase by by marking root by a label "c" (see $E''_1$ the figure 7).

![Fig. 7: Initial set $E''_1$ of the second phase](image)

In the second phase we tried to go down and selected the candidate (labeled by "p") if possible: $E_2 = R^*_2(E''_1) = Election(E''_1)$.

From the result of the second phase $R^*(E''_1)$, we are interested in tree having a path only "c" from root to leaf. The filtration is computed by $E'_1 = Inv(E_2)$.

EFTRSs is described below:

$$O(a(x, y)) \rightarrow a(x, y) \text{ with } a \in \{n, p\}$$

$$Inv(\bot) \rightarrow \bot$$
Functional Term Rewriting Systems

\[ O(\bot) \rightarrow \bot \]
\[ Inv(c(x, y)) \rightarrow c(O(x), Inv(y)) \]
\[ O(\bot) \rightarrow Inv(c(x, y)) \rightarrow c(O(x), O(y)) \]

This trees set \( E'_2 \) is depicted in the figure 8.

We also show how we can find this results in TRS: We can count the number of "c" in the leaves, etc.

We have another solution: We begin from the bottom, replace each label "c" by "o" and go back step by step. The sub tree that has a root labeled "o" is the result we need but now it is tree having a path where there are only the labels "o" from root to leaf.

\begin{align*}
TRS: & \quad c(\bot, \bot) \rightarrow o(\bot, \bot) \\
& \quad c(o(x, y), z) \rightarrow o(o(x, y), z) \\
& \quad o(x, o(y, z)) \rightarrow o(x, o(y, z))
\end{align*}

\begin{align*}
FTRS: & \quad \text{Generated rules} \quad O(o(x, y)) \rightarrow o(x, y) \\
& \quad O(\bot) \rightarrow \bot \\
& \quad H(c(x, y)) \rightarrow o(O(x), y) \\
& \quad H(c(x, y)) \rightarrow o(x, O(y)) \\
& \quad \text{Circulation} \quad H(c(x, y)) \rightarrow c(H(x), y) \\
& \quad H(c(x, y)) \rightarrow c(x, H(y)) \\
& \quad \text{Fixed point} \quad Inv(x) \rightarrow x \\
& \quad Inv(x) \rightarrow Inv(H(x))
\end{align*}

Finally, we consider the sub tree whose root is labeled by "o".

\[ \mathcal{F}^2 = (E'_2 \cap \{o(t_1, t_2) | t_1, t_2 \in T(F_{bin})\}) = O(E'_{2}). \]

This tree is exactly liked the set of trees in the figure 8 except that the path of labels "c" is replaced by the path of labels "o".

\[ \text{Fig. 8: } E_2 = Inv(Election(E'^1_2)) \]

B.3 Experimental results of TRSs, FTRSs and EFTRSs

In this section, we aim at comparing our EFTRS Model-checker (written in JAVA) with two available tools like timbuk ([30], downloadable from http://www.irisa.fr/celtique/genet/timbuk/), maude([26], downloadable from http://maude.cs.uiuc.edu), etc by testing on some communication protocols [1, 37] like Percolate Protocol, Leader Election Protocol and Tree Arbiter Protocol.
Hereafter reported results were obtained on a PC 1.7GHz 1GB RAM. More details of the representations of tests in Timbuk, Maude can be found in Section C.

1. **Tree Arbiter Protocol.** These models are described in the paper. We begin by compute $R^*(v)$ without the requests rules.

In the below table the computation time is computed by second. It should be noted that on the table the exponential increase of the model size $N$ leads to the dramatical increase of reachability configuration number $#\text{Confs}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$#\text{Confs}$</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>43264</td>
<td>56430</td>
<td>3.672</td>
<td>N/A</td>
</tr>
<tr>
<td>$2^{10}$</td>
<td>-</td>
<td>$&gt;3h$</td>
<td>$&gt;3h$</td>
<td>N/A</td>
</tr>
</tbody>
</table>

TRSs

N/A means that our EFTRS model-checker does not support for TRSs and FTRSs.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$#\text{Confs}$</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>43264</td>
<td>$&gt;3h$</td>
<td>82373</td>
<td>N/A</td>
</tr>
<tr>
<td>$2^{10}$</td>
<td>-</td>
<td>-</td>
<td>$&gt;3h$</td>
<td>N/A</td>
</tr>
</tbody>
</table>

FTRSs

Maude is efficient in either TRS or FTRSs while Timbuk cannot realize FTRSs and EFTRSs.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$#\text{Confs}$</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>43264</td>
<td>$&gt;3h$</td>
<td>379816</td>
<td>0.139</td>
</tr>
<tr>
<td>$2^{10}$</td>
<td>4.758 E+16</td>
<td>-</td>
<td>$&gt;3h$</td>
<td>0.184</td>
</tr>
<tr>
<td>$2^{20}$</td>
<td>2.173 E+63</td>
<td>-</td>
<td>-</td>
<td>0.218</td>
</tr>
<tr>
<td>$2^{30}$</td>
<td>8.591 E+383</td>
<td>-</td>
<td>-</td>
<td>0.308</td>
</tr>
<tr>
<td>$2^{35}$</td>
<td>1.929 E+3409</td>
<td>-</td>
<td>-</td>
<td>0.634</td>
</tr>
<tr>
<td>$2^{40}$</td>
<td>8.648 E+9444</td>
<td>-</td>
<td>-</td>
<td>1.075</td>
</tr>
<tr>
<td>$2^{45}$</td>
<td>7.732 E+18490</td>
<td>-</td>
<td>-</td>
<td>1.418</td>
</tr>
<tr>
<td>$2^{50}$</td>
<td>1.379 E+30547</td>
<td>-</td>
<td>-</td>
<td>1.809</td>
</tr>
</tbody>
</table>

EFTRSs

Maude for EFTRSs are broken very soon while EFTRSs-checker latency remains quasi constant for this model. It can be explained by the speeding up of LFP approach integrated in our EFTRSs-checker.

In the other hand, we also have Benchmarks of the model with the requests rules.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$#\text{Confs}$</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>8.539 E+8</td>
<td>$&gt;3h$</td>
<td>$&gt;3h$</td>
<td>N/A</td>
</tr>
<tr>
<td>$2^{10}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>N/A</td>
</tr>
</tbody>
</table>

TRSs
This problem now becomes so difficult that neither Maude nor Timbuk can reach to the model size $N = 2^5$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>$8.539 \times 10^8$</td>
<td>$&gt; 3h$</td>
<td>$&gt; 3h$</td>
<td>N/A</td>
</tr>
</tbody>
</table>

| $2^{10}$ | - | - | - | N/A |

FTRSs

But LFP technique still brings to our checker a surprise performance\(^3\) to reach to the model size $N = 2^{400}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>$8.539 \times 10^8$</td>
<td>$&gt; 3h$</td>
<td>$&gt; 3h$</td>
<td>0.109</td>
</tr>
</tbody>
</table>

| $2^{10}$ | $1.989 \times 10^{273}$ | - | - | 0.196 |

| $2^{20}$ | $5.350 \times 10^{278807}$ | - | - | 0.256 |

| $2^{50}$ | ? | - | - | 0.430 |

| $2^{100}$ | ? | - | - | 0.789 |

| $2^{200}$ | ? | - | - | 1.453 |

| $2^{500}$ | ? | - | - | 2.137 |

| $2^{1000}$ | ? | - | - | 2.866 |

EFTRSs

2. **Percolate Protocol.** Their descriptions and models occur in Section B.1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>11047</td>
<td>0.138</td>
<td>1.479</td>
<td>N/A</td>
</tr>
</tbody>
</table>

| $2^{10}$ | $8.445 \times 10^{135}$ | 24.179 | $> 3h$ | N/A |

| $2^{20}$ | - | $> 3h$ | - | N/A |

TRSs

For this example, Timbuk has alway the best performance in either TRSs or FTRSs.

<table>
<thead>
<tr>
<th>$N$</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>11047</td>
<td>0.380</td>
<td>6.804</td>
<td>N/A</td>
</tr>
</tbody>
</table>

| $2^{10}$ | $8.445 \times 10^{135}$ | 2.956 | $> 3h$ | N/A |

| $2^{20}$ | - | $> 3h$ | - | N/A |

FTRSs

<table>
<thead>
<tr>
<th>$N$</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^5$</td>
<td>11047</td>
<td>0.165</td>
<td>19.041</td>
<td>0.018</td>
</tr>
</tbody>
</table>

| $2^{10}$ | $8.445 \times 10^{135}$ | 0.555 | $> 3h$ | 0.053 |

| $2^{20}$ | $4.508 \times 10^{139402}$ | 248.830 | - | 0.102 |

| $2^{50}$ | ? | $> 1G$ | - | 0.139 |

| $2^{100}$ | ? | - | - | 0.183 |

| $2^{200}$ | ? | - | - | 0.293 |

| $2^{500}$ | ? | - | - | 0.396 |

\(^3\) The configurations count is noted "?" when the library *java.math.BigDecimal* we used can not return the result within three hours.
EFTRSs

Like the Tree Arbiter Protocol, EFTRSs-checker latency also remains quasi constant while the others are broken very soon. It can be explained by the speed up of LFP approach for EFTRSs.

3. Leader Election Protocol. Their descriptions and models occur in Section B.1. The performance of the LFP approach is shown on the table below with the Election Leader Protocol. Notice that in the worst case, the problem size \(2^n\) will produce \(2^{(n-1)}\) election possibilities. N/A in the below table means that our EFTRS model-checker does not support for TRSs.

<table>
<thead>
<tr>
<th>N</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^5)</td>
<td>512</td>
<td>7,595</td>
<td>71.981</td>
<td>N/A</td>
</tr>
<tr>
<td>(2^{10})</td>
<td>-</td>
<td>&gt; 3h</td>
<td>&gt;3h</td>
<td>N/A</td>
</tr>
</tbody>
</table>

TRSs

Like the Tree Arbiter Protocol, the tables also show that Timbuk tool is efficient in TRS but it cannot realize FTRSs and EFTRSs as Maude.

<table>
<thead>
<tr>
<th>N</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^5)</td>
<td>16</td>
<td>&gt; 3h</td>
<td>0.457</td>
<td>N/A</td>
</tr>
<tr>
<td>(2^{10})</td>
<td>512</td>
<td>-</td>
<td>&gt; 3h</td>
<td>N/A</td>
</tr>
<tr>
<td>(2^{20})</td>
<td>5.243 E+5</td>
<td>-</td>
<td>-</td>
<td>N/A</td>
</tr>
</tbody>
</table>

FTRSs

<table>
<thead>
<tr>
<th>N</th>
<th>#Confs</th>
<th>Timbuk</th>
<th>Maude</th>
<th>EFTRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^5)</td>
<td>16</td>
<td>&gt; 3h</td>
<td>0.920</td>
<td>0.093</td>
</tr>
<tr>
<td>(2^{10})</td>
<td>512</td>
<td>-</td>
<td>&gt; 3h</td>
<td>0.159</td>
</tr>
<tr>
<td>(2^{20})</td>
<td>5,243 E+5</td>
<td>-</td>
<td>-</td>
<td>0.256</td>
</tr>
<tr>
<td>(2^{40})</td>
<td>5,629 E+14</td>
<td>-</td>
<td>-</td>
<td>0.956</td>
</tr>
<tr>
<td>(2^{60})</td>
<td>6,338 E+29</td>
<td>-</td>
<td>-</td>
<td>3.194</td>
</tr>
<tr>
<td>(2^{80})</td>
<td>8,035 E+59</td>
<td>-</td>
<td>-</td>
<td>18.355</td>
</tr>
<tr>
<td>(2^{100})</td>
<td>1,019 E+90</td>
<td>-</td>
<td>-</td>
<td>57.430</td>
</tr>
<tr>
<td>(2^{120})</td>
<td>1,291 E+120</td>
<td>-</td>
<td>-</td>
<td>138.699</td>
</tr>
</tbody>
</table>

EFTRSs

The tables also show that our EFTRS model-checker is always efficient than Maude.

C Timbuk, Maude tests

This section we show how we can build the tests in Timbuk, Maude. Here is the representations of tests in Timbuk, Maude:

- Timbuk There is no problem in classic TRSs but we must add the equations \(H(H(X))=H(X)\) to ensure the termination of the fixpoint in FTRSs.
- Maude There is no problem in classic TRSs but if we want to use the fixpoint in FTRSs, we should define \(F_{bin}\) and \(F_{NT}\) to ensure the termination.
C.1 Timbuk

1. Classical TRS.
2. **FTRS with Timbuk.** Add the equations \( Rules \ H(H(X)) = H(X) \) to ensure the termination of the fixpoint.

\[
\begin{align*}
F(X) & \rightarrow X \\
F(X) & \rightarrow F(H(X))
\end{align*}
\]

3. **EFTRS with Timbuk.** Add the equations \( Rules \ H(H(X)) = H(X) \) to ensure the termination of the fixpoint.

\[
\begin{align*}
F(b(X,Y)) & \rightarrow b(X,Y) \\
F(t(X,Y)) & \rightarrow t(X,Y) \\
F(b(X,Y)) & \rightarrow F(H(b(X,Y))) \\
F(t(X,Y)) & \rightarrow F(H(t(X,Y)))
\end{align*}
\]

Further details see at http://eftrs.svn.sourceforge.net/viewvc/eftrs/timbuk/

C.2 Maude

\[
\text{load arbiter.frs.maude}$
\text{search F(t(r(T,T),r(T,T))) =>* t(X:Terminal, Y:Terminal) .}
\]

1. Classical TRS.
2. **FTRS with Maude.** We should distinguish reducible and irreducible symbols to ensure the termination.

\[
\begin{align*}
rl F(X) & \rightarrow X \\
rl F(X) & \rightarrow F(H(X))
\end{align*}
\]

3. **EFTRS with Maude.** We should distinguish reducible and irreducible symbols to ensure the termination.

\[
\begin{align*}
rl F(b(X,Y)) & \rightarrow b(X,Y) \\
rl F(t(X,Y)) & \rightarrow t(X,Y) \\
rl F(b(X,Y)) & \rightarrow F(H(b(X,Y))) \\
rl F(t(X,Y)) & \rightarrow F(H(t(X,Y)))
\end{align*}
\]

Further details see at http://eftrs.svn.sourceforge.net/viewvc/eftrs/maude/