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Rapport de Recherche

A new bound for parsimonious edge-colouring of graphs with maximum degree three

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A NEW BOUND FOR PARSIMONIOUS EDGE-COLOURING OF GRAPHS WITH MAXIMUM DEGREE THREE

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Abstract. In a graph G of maximum degree 3, let $\gamma(G)$ denote the largest fraction of edges that can be 3 edge-coloured. Rizzi [9] showed that $\gamma(G) \geq 1 - \frac{2}{3g_{odd}(G)}$ where $g_{odd}(G)$ is the odd girth of G, when G is triangle-free. In [3] we extended that result to graph with maximum degree 3. We show here that $\gamma(G) \geq 1 - \frac{2}{3g_{odd}(G)+2}$, which leads to $\gamma(G) \geq \frac{15}{17}$ when considering graphs with odd girth at least 5, distinct from the Petersen graph.

1. Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. Staton [10] (and independently Locke [8]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [7]) with at least $\frac{7}{9}$ of the edges of G. Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ (where $\gamma(G)$ denote the largest fraction of edges of G that can be 3 edge-coloured) while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Steffen [11] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen graph. In [3], we extended this result to graphs with maximum degree 3 where bridges are allowed. With the exception of G_5 (a C_5 with two chords), the graph P' obtained from two copies of G_5 by joining by an edge the two vertices of degree 2 and the Petersen graph, every graph G is such that $\gamma(G) > \frac{13}{15}$. Rizzi [9] showed that $\gamma(G) \geq 1 - \frac{2}{3g_{odd}(G)}$ where $g_{odd}(G)$ is the odd girth of G, when G is triangle-free. In [3] we extended that result to graph with maximum degree 3 (triangles are allowed). We show here that $\gamma(G) \geq 1 - \frac{2}{3g_{odd}(G)+2}$, which leads to $\gamma \geq \frac{15}{17}$ when considering graphs with odd girth at least 5, distinct from the Petersen graph.

Theorem 1.1. *let G be a graph with maximum degree 3 distinct from the Petersen graph. Then $\gamma(G) \geq 1 - \frac{2}{3g_{odd}(G)+2}$*

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2. Technical lemmas

Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G. It is often of interest to try to use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that ϕ is $\delta - minimum$. Since any two $\delta-$minimum edge-colouring of G have the same number of edges coloured δ we shall denote by $s(G)$ this number. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$, $x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours (this subgraph being a union of paths and even cycles where the colours x and y alternate) and $E_\phi(x)$ is the set of edges coloured with x.

In [5] we gave without proof (in French, see [4] for a translation) results on δ-minimum edge-colourings of cubic graphs

Lemma 2.1. [5, 6, 4] *Let ϕ be a $\delta-$minimum edge-colouring of G. For any edge $e = uv$ coloured with δ there are two colours x and y in $\{\alpha, \beta, \gamma\}$such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining these two ends. Moreover e has one end of degree 2 and the other of degree 3 or the two ends of degree 3*

Remark 2.2. An edge coloured with δ by the $\delta-$minimum edge-colouring ϕ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. It is clear that A_ϕ, B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge of coloured with δ with one end of degree 2 is contained in 2 such sets. Assume indeed that $e = uv$ is coloured with δ while $d(u) = 3$ and $d(v) = 2$ then, if u is incident to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$. When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ. In the following lemma we consider an edge in A_ϕ, an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 2.3. [5, 6, 4] *Let ϕ be a $\delta-$minimum edge-colouring of G and let e be an edge in A_ϕ then for any edge $e' \in C_{A_\phi}(e)$ there is a $\delta-$minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ, ϕ and ϕ' only differ on the edges of $C_{A_\phi}(e)$.*

For each edge $e \in E_\phi(\delta)$ (where ϕ is a $\delta-$minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ, B_ϕ or C_ϕ. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 2.4. [5, 6, 4] *Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are (vertex) disjoint.*

Lemma 2.5. [5, 6, 4] *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then v_1 is the only vertex in $N(u)$ of degree 2 and for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

Lemma 2.6. [5, 6, 4] *Let e_1 and e_2 be two edges of $E_\phi(\delta)$. If e_1 and e_2 are contained in two distinct sets of A_ϕ, B_ϕ or C_ϕ then $\{e_1, e_2\}$ induces a $2K_2$ otherwise e_1, e_2 are joined by at most one edge.*

Lemma 2.7. [5, 6, 4] *Let e_1, e_2 and e_3 be three distinct edges of $E_\phi(\delta)$ contained in the same set A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2, e_3\}$ induces a subgraph with at most four edges.*

Lemma 2.8. [3] *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

A cubic graph G on n vertices is called a *permutation graph* if G has a perfect matching M such $G-M$ is the union of two chordless cycles A and B of equal length $\frac{n}{2}$. When we delete one edge of the perfect matching M in the above permutation graph G, we shall say that the graph obtained is a *near-permutation graph*.

Lemma 2.9. *Let G be a permutation graph or a near-permutation graph with n vertices and odd girth $\frac{n}{2}$. Suppose that G is not 3-edge colourable. Then G is the Petersen graph or the Petersen graph minus one edge.*

Proof Obviously, since G is not 3-edge colourable, $\frac{n}{2}$ is certainly odd. Let $A = a_0a_1 \dots a_{2k}$ and $B = b_0b_1 \dots b_{2k}$ be the two chordless cycles of length $\frac{n}{2} = 2k+1$ which partition $V(G)$. When $2k+1 = 3$, it can be easily verified that G is 3−edge colourable and when $2k+1 = 5$ G is the Petersen graph or the Petersen graph minus one edge. Assume thus that $2k+1 \geq 7$.

Since at most one vertex of A and one vertex of B have degree 2, we can suppose that a_0, a_1, a_2 are joined to 3 distinct vertices of B. Without loss of generality we suppose that $a_0b_0 \in E(G)$. Since G is not 3−edge colourable a_1b_1 and a_1b_{2k} are not edges of G and since the odd girth is at least 7 we do not have the edges a_1b_2 and a_1b_{2k-1}. Henceforth let b_i $(2 < i \leq 2k-2)$ the neighbour of a_1. One of the two paths determined by b_0 and b_i on B must have odd length. Suppose, without loss of generality, that $b_0b_1 \dots b_i$ has odd length then we have an odd cycle $a_0b_0b_1 \dots b_ia_1$ whose length is at least $2k+1$, which leads to $i = 2k-2$.

The vertex a_2 is not joined to b_{2k-1} or b_{2k-3}, otherwise G is 3-edge colourable, neither to b_{2k} ,b_{2k-4} or b_1, otherwise the odd girth is 5. Henceforth a_2 is joined to some vertex b_j with $2 \leq j \leq 2k-5$. If j is odd then $b_0b_1 \dots b_ja_2a_1a_0$ is an odd cycle of length at most $2k-1$, impossible. If j is even then $b_j \dots b_{2k-3}b_{2k-2}a_1a_2$ an odd cycle of length at most $2k-1$, impossible. □

3. Proof of Theorem 1.1

Proof

Let ϕ be a $\delta-$minimum edge-colouring of G and $E_\phi(\delta) = \{e_1, e_2 \dots e_{s(G)}\}$. $\mathcal{C}$ being the set of odd cycles associated to edges in $E_\phi(\delta)$, we write $\mathcal{C} = \{C_1, C_2 \dots C_{s(G)}\}$ and suppose that for $i = 1, 2 \dots s(G)$, e_i is an edge of C_i. We know by Lemma 2.4 that the cycles of $\mathcal{C}$ are vertex-disjoint.

Let $l(\mathcal{C}) = \sum_{C \in \mathcal{C}} l(C)$ (where $l(C)$ is the length of the cycle C) and assume that ϕ has been chosen in such a way that $l(\mathcal{C})$ is maximum.

Let us write $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$, where $\mathcal{C}_2$ denotes the set of odd cycles of $\mathcal{C}$ which have a vertex of degree 2, while $\mathcal{C}_3$ is for the set of cycles in $\mathcal{C}$ whose all vertices have degree 3. Let $k = |\mathcal{C}_2|$, obviously we have $0 \leq k \leq s(G)$ and $\mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset$.

If $C_i \in \mathcal{C}_2$, we may suppose that e_i has a vertex of degree 2 (see Lemma 2.3) and we can associate to e_i another odd cycle say $C_i^{'}$ (Remark 2.2) whose edges distinct from e_i form an even path P_i using at least $\frac{g_{odd}(G)}{2}$ edges which are not edges of C_i. When $l(C_i) = g_{odd}(G)$, C_i has no chord and it is an easy task to find a supplementary edge of P_i not belonging to C_i. When $l(C_i) \geq g_{odd}(G) + 2$ (recall that C_i has odd length), we can choose an edge of C_i as a *private* edge. In both cases $C_i \cup C_i^{'}$ contains at least $\frac{3}{2}g_{odd}(G) + 1$ edges. Consequently there are at least $k \times (\frac{3}{2}g_{odd}(G) + 1)$ edges in $\bigcup_{C_i \in \mathcal{C}_2} (C_i \cup C_i^{'})$.

When $C_i \in \mathcal{C}_3$, C_i contains at least $g_{odd}(G)$ edges, moreover, each vertex of C_i being of degree 3, there are $\frac{s(G)-k}{2} \times g_{odd}(G)$ additional edges which are incident to a vertex of $\bigcup_{C_i \in \mathcal{C}_3} C_i$. Let us remark that each above additional edge is counted as $\frac{1}{2}$ whatever are these edges. In order to refine our counting, we need to introduce the following notion of *free* edge. An edge will be said to be *free* when at most one end belongs to some $C_i \in \mathcal{C}_3$.

Suppose that we can associate to each $C_i \in \mathcal{C}_3$ one private edge or two private free edges. Since $C_i \cap C_j = \emptyset$ and $C_i^{'} \cap C_j = \emptyset$ $(1 \leq i, j \leq s(G),\ i \neq j)$, we would have

$$m \geq (k \times (\frac{3}{2}g_{odd}(G) + 1) + (s(G) - k) \times (g_{odd}(G) + 1) + \frac{s(G) - k}{2} \times (g_{odd}(G))$$

and

$$m \geq s(G) \times (\frac{3}{2} \times g_{odd}(G) + 1).$$

Consequently $\gamma(G) = 1 - \frac{s(G)}{m} \geq 1 - \frac{2}{3g_{odd}(G)+2}$, as claimed.

Our goal now is to associate to each $C_i \in \mathcal{C}_3$ one private edge or two private free edges.

When C_i is incident to at least two free edges, let us choose any two such free edges as the private free edges associate to C_i. By definition, these two free edges are not incident to any $C_j \in \mathcal{C}_3$ with $j \neq i$, insuring thus that they cannot be associated to C_j . When $l(C_i) \geq g_{odd}(G) + 2$, we choose any edge of C_i as a private edge.

Assume thus that $l(C_i) = g_{odd}(G)$. Hence $C_i = x_0x_1 \ldots x_{g_{odd}(G)-1}$ is chordless. Suppose that C_i is incident to at most one free edge. Without loss of generality, we can consider that x_0 is the only possible vertex of C_i incident to some free edge. Since the edge incident to x_1, not belonging to C_i, is not free, let $C_j \in \mathcal{C}_3$,$i \neq j$, such that x_1 is adjacent to $y_1 \in C_j$. In the same way, the edge incident to x_2, not belonging to C_i, is not free. Let $C_{j'} \in \mathcal{C}_3$,$i \neq j^{'}$, such that x_2 is adjacent to $z_2 \in C_{j'}$. Suppose that $j \neq j^{'}$, then by Lemma 2.3 we can consider that x_1x_2 is coloured δ by ϕ as well as one of the edges of C_j incident with the vertex y_1 and one of the edges of $C_{j'}$ incident with z_2, a contradiction with Lemma 2.7 or Lemma 2.6. Henceforth, x_2 is adjacent to some vertex of C_j. In the same way every vertex of C_i, distinct from x_0, is adjacent to some vertex of C_j.

In this situation, let us say that C_i is *extremal for* C_j. Assume that we have a set of $p \geq 2$ distinct extremal cycles of $\mathcal{C}_3$ for C_j. Since two distinct extremal cycles have no consecutive neighbours in C_j (otherwise we get a contradiction with Lemmas 2.7 or 2.6 as above), C_j has length at least $p \times (g_{odd}(G) - 1) + 2$. But $p \times (g_{odd}(G) - 1) + 2 \geq g_{odd}(G) + p + 1$ as soon as $g_{odd}(G) \geq 3$. hence, when $p \geq 2$, we can associate to each extremal cycle for C_j a private edge belonging to C_j as well as a private edge for C_j itself, since $l(C_j) \geq g_{odd}(G) + p + 1$.

Assume thus that C_i is the only extremal cycle for C_j. If $l(C_j) \geq g_{odd}(G) + 2$, we can associate any edge of C_j as a private edge of C_i and any other edge of C_j as a private edge of C_j. It remains thus to consider the case where $l(C_j) = g_{odd}(G)$. In that situation, C_i is an extremal cycle for C_j, as well as C_j is an extremal cycle for C_i and the subgraph induced by $C_i \cup C_j$ is a permutation graph or a near permutation graph with odd girth $g_{odd}(G) = \frac{|C_i \cup C_j|}{2}$.

By Lemma 2.9 G itself is the Petersen graph or $C_i \cup C_j$ induces a Petersen graph minus one edge (by the way, $g_{odd}(G) = 5$). By hypothesis, the first case is excluded. Assume thus that $C_i \cup C_j$ induces a Petersen graph minus one edge. In the last part of this proof, we show that this situation is not possible.

In order to fix the situation let H be the subgraph of G not containing $C_i \cup C_j$. We suppose that C_j is the chordless cycle of length 5 $y_0y_3y_1y_4y_2$ while x_1y_1, x_2y_2, x_3y_3 and x_4y_4 are the edges joining C_i to C_j. Moreover x_0 is joined to some vertex $a \in V(H)$ and y_0 is joined to some vertex $b \in V(H)$. Without loss of generality, we can consider that ϕ colours alternately the edges of C_i (C_j respectively) with β and γ with the exception of the edge x_0x_1 coloured with δ (y_0y_3 respectively).

The edges x_1y_1, x_2y_2, x_3y_3 and x_4y_4 are thus coloured with α as well as the edges x_0a and y_0b (let us remark that $a \neq b$). The final situation is depicted in Figure 1.

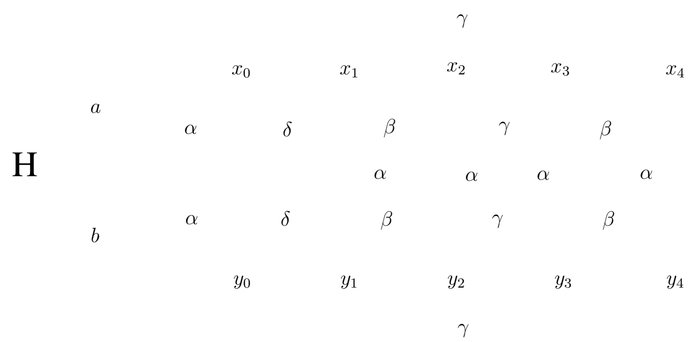

FIGURE 1. Final situation

Without changing any colour in H and keeping the colour α for the edges x_0a and y_0b, we can construct a new $\delta-$minimum edge-colouring ϕ' in the following way (see Figure 2):

- x_0x_4 and x_1x_2 are coloured with δ
- x_4y_4, y_0y_2, x_2x_3 and y_1y_3 are coloured with β
- x_1y_1, x_3y_3 and y_4y_2 are coloured with α
- $x_0x_1, x_3x_4, y_0y_3, y_1y_4$ and x_2y_2 are coloured with γ.

Let $\mathcal{C}'$ be the set of odd cycles associated to edges coloured with δ by ϕ'. Since we do not have change any colour in H, we have $\mathcal{C}' = \mathcal{C} - \{C_i, C_j\} + \{C_i', C_j'\}$ where C_i' is the chordless cycle of length 5 $x_1x_2x_3y_3y_1$ and C_j' is the odd cycle obtained by

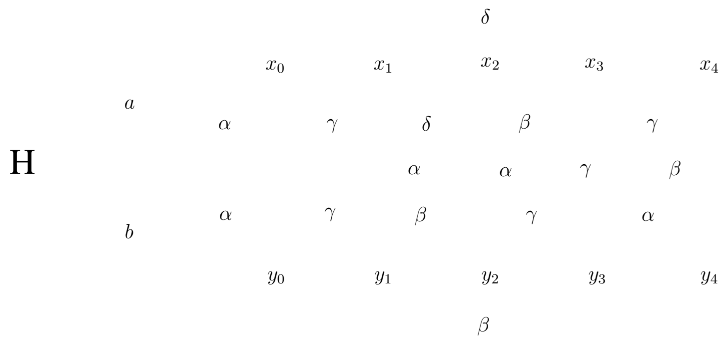

FIGURE 2. New colouring $\phi^{'}$

concatenation of the path $ax_0x_4y_4y_2y_0b$ and the path coloured alternately β and α joining a to b in H (dashed line in Figure 2), whose existence comes from Lemma 2.1. Since the length of $C_j^{'}$ is at least 7, $l(\mathcal{C}^{'}) > l(\mathcal{C})$, a contradiction with the initial choice of ϕ.

□

4. APPLICATION TO BRIDGELESS CUBIC GRAPHS

In this section we show that for any bridgeless cubic graph G distinct from the Petersen graph we have $\gamma(G) \geq \frac{15}{17}$.

A *triangle* $T = \{a, b, c\}$ is said to be *reducible* whenever its neighbours are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 by shrinking this triangle into a single vertex and joining this new vertex to the neighbours of T in G.

Lemma 4.1. [1] *Let G be a graph with maximum degree 3. Assume that $T = \{a, b, c\}$ is a reducible triangle and let G' be the graph obtained by reduction of this triangle. Then $\gamma(G) > \gamma(G')$.*

Let P_{12} be the cubic graph obtained from the Petersen graph by replacing a vertex by a triangle. One can easily verify that $s(P_{12}) = 2$ leading immediately to

Lemma 4.2. $\gamma(P_{12}) = \frac{8}{9}$.

Let D_k ($k \geq 1$) be the graph depicted in Figure 3 (where a_k and b_k are not adjacent) and let G be a graph of maximum degree 3 containing a subgraph H isomorphic to D_k for some $k \geq 1$. If we delete the vertices of H distinct from a_k and b_k and add a new edge between these two vertices we get a new graph $G^{'}$ with maximum degree 3. Let us say that $G^{'}$ is obtained from G by *reducing* D_k

Lemma 4.3. *Let G be a graph with maximum degree 3. Assume that G contains a subgraph isomorphic to D_k for some $k \geq 1$ and let G' be the graph obtained from G by reducing D_k. Then $s(G) = s(G^{'})$ and $\gamma(G) > \gamma(G')$.*

Proof Let ϕ be a $\delta-$minimum edge-colouring of G. If $\phi(a_ka_{k-1}) = \phi(b_kb_{k-1})$, then we get an immediate 3−edge colouring of $G^{'}$ by giving this common colour to the edge a_kb_k which gives $s(G^{'}) \leq s(G)$. If $\phi(a_ka_{k-1}) \neq \phi(b_kb_{k-1})$, then one can easily verify that one edge at least of D_k must be coloured with δ. Hence we can obtain a 3−edge colouring of $G^{'}$ by giving the colour δ to a_kb_k which gives $s(G^{'}) \leq s(G)$ leading also to $s(G^{'}) \leq s(G)$. Conversely, one can easily extend a $\delta-$minimum edge-colouring of $G^{'}$ to a 3−edge colouring of G using at most $S(G^{'})$ edges coloured with δ. Hence $s(G) = s(G^{'})$ as claimed.

Since $|E(G)| > |E(G^{'})|$, we have, by Lemma 2.8,

$$\gamma(G) = 1 - \frac{s(G)}{|E(G)|} > 1 - \frac{s(G^{'})}{|E(G')|} = \gamma(G^{'})$$

. □

Lemma 4.4. *Let G be a bridgeless cubic graph and suppose that G contains a subgraph isomorphic to D_k for some $k \geq 1$. Let G' be the graph obtained from G by reducing D_k and assume that $G^{'}$ is isomorphic to the Petersen graph. Then* $\gamma(G) > \frac{15}{17}$.

Proof Obviously, G contains at least 6 edges more than the Petersen graph. Since $\gamma(G) = 1 - \frac{s(G)}{|E(G)|}$ by Lemma 2.8 and $s(G) = 2$ by Lemma 4.3, we have immediately $\gamma(G) \geq 1 - \frac{2}{15+6} > \frac{15}{17}$, as claimed. □

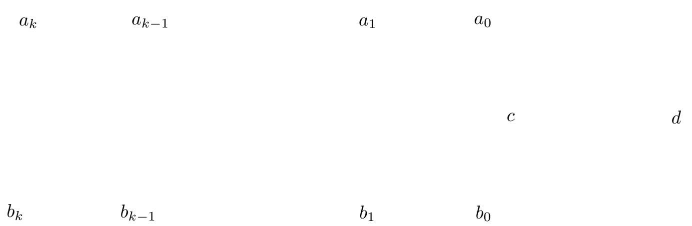

FIGURE 3. D_k

Theorem 4.5. *Let G be a bridgeless cubic graph distinct from the Petersen graph. Then* $\gamma(G) \geq \frac{15}{17}$

Proof When G has at most 10 vertices, it is well known that either G is 3−edge colourable or G is isomorphic to the Petersen graph. The latter case is is excluded by the hypothesis. When G has 12 vertices, the only bridgeless non 3−edge colourable cubic graph is P_{12} for which the result is true by Lemma 4.4. Hence, for bridgeless cubic graph with at most 12 vertices the result holds true. Assume by induction that every bridgeless cubic graph H with at most $n \geq 12$ vertices is such that $\gamma(H) \geq \frac{15}{17}$ and let us prove the result for a bridgeless cubic graphs G with $n + 2$ vertices.

If $g_{odd}(G) \geq 5$ the result comes from Theorem 1.1. We can thus suppose that G contains a triangle T. If this triangle is reducible, the result follows from Lemma 4.1. When T is not reducible, that means that either G is reduced to K_4 (the three neighbours of T are reduced to a single vertex) or G contains a subgraph isomorphic to D_k for some $k \geq 1$. The former case is impossible since G has at least 14 vertices. In the latter case, we use Lemma 4.3 or Lemma 4.4 to conclude. □

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