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On minimum (K_q, k) stable graphs

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ON MINIMUM (K_q, k) STABLE GRAPHS

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Abstract. A graph G is a (K_q, k) vertex stable graph ($q \geq 3$) if it contains a K_q after deleting any subset of k vertices ($k \geq 0$). We are interested by the $(K_q, \kappa(q))$ vertex stable graphs of minimum size where $\kappa(q)$ is the maximum value for which for every nonnegative integer $k < \kappa(q)$ the only (K_q, k) vertex stable graph of minimum size is K_{q+k}.

1. Introduction

In [6] Horwárth and Katona consider the notion of (H, k) *stable* graph: given a simple graph H, an integer k and a graph G containing H as subgraph, G is a *a* (H, k) *stable graph* whenever the deletion of any set of k edges does not lead to a H-free graph. These authors consider (P_n, k) stable graphs and prove a conjecture stated in [5] on the minimum size of a (P_4, k) stable graph. In [2], Dudek, Szymański and Zwonek are interested in a vertex version of this notion and introduce the (H, k) vertex stable graphs. In [4] we have characterized (K_q, k) vertex stable graphs with minimum size for $q = 3, 4, 5$ (where K_q denotes the clique on q vertices).

Definition 1.1. [2] Given an integer $k \geq 0$ and a graph H, a graph G containing a subgraph isomorphic to H is said to be *a* (H, k) *vertex stable graph* if, for every subset S of k vertices, $G - S$ contains (a subgraph isomorphic to) H.

In this paper, we are only interested by (H, k) vertex stable graphs and, since no confusion will be possible, a (H, k) vertex stable graph shall be simply called a (H, k) *stable graph.*

Definition 1.2. A (H, k) stable graph with minimum size (i.e. with minimum number of edges) is called *minimum* (H, k) *stable graph.* The minimum size of a (H, k) stable graph shall be denoted by $stab(H, k)$.

It is clear that if G is a (H, k) stable graph with minimum size then the graph obtained from G by addition or deletion of some isolated vertices is also minimum (H, k) stable. Hence we shall asume that all the graphs considered in the paper have no isolated vertices.
Here we consider (K_q, k) stable graphs. We have proved in [4] that $stab(K_q, k) = \binom{q+k}{2}$ for $q \geq 3$ and $k = 1, 2$. Moreover K_q is the only minimum $(K_q, 0)$ stable graph for $q \geq 2$ and K_{q+1} is the unique $(K_q, 1)$ stable graph for $q \geq 4$. Dudek, Szymański and Zwonek have proved the following result.

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Theorem 1.3. [2] *For every $q \geq 4$, there exists an integer $k(q)$ such that $stab(K_q, k) \leq (2q-3)(k+1)$ for $k \geq k(q)$.*

As a consequence of this last result, for every $k \geq k(q)$ the graph K_{q+k} is not minimum (K_q, k) stable.

Definition 1.4. For every integer $q \geq 4$, we denote by $\kappa(q)$ the greatest integer such that for $1 \leq k < \kappa(q)$ the only minimum (K_q, k) stable graph is K_{q+k}.

In two previous papers we have proved the following.

Theorem 1.5. [3, 4] *$\kappa(3) = 1$, $\kappa(4) = 3$, $\kappa(5) = 4$ and for $q \geq 6$ $\kappa(q) > \frac{q}{2} + 1$.*

In this paper we give an upper bound for the value of $\kappa(q)$.

Theorem 1.6. *For every $q \geq 4$, if $\kappa(q)$ is even then $\kappa(q) < \sqrt{2(q-1)(q-2)}$ and if $\kappa(q)$ is odd then $\kappa(q) < \sqrt{1 + 2(q-1)(q-2)}$*

We prove that these upper bounds are reached for values of q such that there exists a minimum $(K_q, \kappa(q))$ stable disconnected graph (note that it is the case for $q = 4$ and $q = 5$).

Theorem 1.7. *Let $q \geq 4$ and suppose that there exists a disconnected minimum $(K_q, \kappa(q))$ stable graph. Set $\rho(q) = \lceil \sqrt{\frac{1}{2}(q-1)(q-2)}\, \rceil - 1$.*
If $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$ then $\kappa(q) = 2\rho(q) + 1$.
If $\frac{1}{2}(q-1)(q-2) \leq \rho(q)^2 + \rho(q)$ then $\kappa(q) = 2\rho(q)$.

Proofs of Theorems 1.6 and 1.7 shall be given in subsection 3.2.

If there is no minimum disconnected $(K_q, \kappa(q))$ stable graph then, by definition of $\kappa(q)$, there exists a connected minimum $(K_q, \kappa(q))$ stable graph G_q which is not complete. We think that it never happens, so we propose the following conjecture.

Conjecture 1.8. *If G is a minimum (K_q, k) stable graph then every component of G is complete.*

If this last conjecture is true then Theorem 1.7 gives the exact value of $\kappa(q)$ for every $q \geq 4$.

2. Notations and general results

For terms not defined here we refer to [1]. As usually, the *order* of a graph G is the number of its vertices and the *size* of G is the number of its edges (it is denoted by $e(G)$). The disjoint union of two graphs G_1 and G_2 is denoted by $G_1 + G_2$. The union of p mutually disjoint copies of a graph G is denoted by pG. For any set A of vertices, we denote by $G[A]$ the subgraph induced by A and by $G - A$ the subgraph induced by $V(G) - A$. If $A = \{v\}$ we write $G - v$ for $G - \{v\}$. For any set F of edges, we denote by $G - F$ the spanning subgraph $(V(G), E(G) - F)$. If $F = \{e\}$ we write $G - e$ instead of $G - \{e\}$. A complete subgraph of order q of G is called a *q-clique* of G. The complete graph of order q is denoted by K_q. When a graph G contains a q-clique as subgraph, we say "G contains a K_q".

Lemma 2.1. [2] *Let G be a (H, k) stable graph with $k \geq 1$. Then, for every vertex v, $G - v$ is $(H, k-1)$ stable.*

A set of vertices of G that intersects every subgraph of G isomorphic to H is called a *transversal of all the subgraphs isomorphic to* H or simply a H*-transversal* of G. A H-transversal of G having the minimum number of vertices is said to be a *minimum* H*-transversal of* G. The number of vertices of a minimum H-transversal is denoted by $\tau_H(G)$. Remark that G is (H, k) stable if and only if $\tau_H(G) \geq k + 1$

Definition 2.2. Let G be a (H, k) stable graph. If G has a minimum H-transversal having exactly $k + 1$ vertices, G is said to be *exactly* (H, k) *stable.*

Remark 2.3. H is the unique $(H, 0)$ stable graph with minimum size.

Lemma 2.4. [2] *Let* G *be a* (H, k) *stable graph with* $k \geq 1$ *and* $e \in E(G)$ *such that* $G - e$ *is not* (H, k) *stable. Then* G *is exactly* (H, k) *stable and* $G - e$ *is exactly* $(H, k - 1)$ *stable.*

Definition 2.5. [2] Let G be a (H, k) stable graph. If $G - e$ is not (H, k) stable for every edge $e \in E(G)$, G is said to be *minimal* (H, k) *stable.*

Remark 2.6. In [2] "minimal (H, k) stable graphs" are called "strong (H, k) stable graphs" by the authors. Note that a (H, k) stable graph G is minimal (H, k) stable if and only if for every $e \in E(G)$ the graph $G - e$ is exactly $(H, k - 1)$ stable. Moreover, a minimal (H, k) stable graph is exactly (H, k) stable.

If there exists an edge e of an (H, k) stable graph G such that there are no subgraph isomorphic to H containing e then $G - e$ is a (H, k) stable graph. Hence, we have the following.

Lemma 2.7. [2] *Every edge of a minimal* (H, k) *stable graph is contained in a subgraph isomorphic to* H*. Consequently, every vertex of a minimal* (H, k) *stable graph is also contained in a subgraph isomorphic to* H*.*

Remark 2.8. Clearly, every minimum (H, k) stable graph is minimal (H, k) stable.

One may ask what happens for components of a (H, k) stable graph. The following theorem gives us an answer.

Theorem 2.9. *Let* G *be an exactly* (H, k) *stable graph, and let* G_1*,* G_2*, ...,* G_r*, with* $r \geq 1$*, be its components. Then, there exist integers* k_1*,* k_2*, ...,* k_r*, with* $0 \leq k_i \leq k$*, sucht that*

i) *for every* i*, with* $1 \leq i \leq r$*,* G_i *is exactly* (H, k_i) *stable,*

ii)

$$\sum_{i=1}^{r} k_i + (r - 1) = k$$

G *is minimal* (H, k) *stable if and only if for every* i*,* $1 \leq i \leq r$*,* G_i *is minimal* (H, k_i) *stable. Moreover, if* G *is minimum* (H, k) *stable then for every* i*,* $1 \leq i \leq r$*,* G_i *is minimum* (H, k_i) *stable.*

Proof For each i, $1 \leq i \leq r$, let us consider a minimum H-transversal of G_i, say T_i, and set $k_i = |T_i| - 1$. Clearly, for each i the graph G_i is exactly (H, k_i) stable and the set $T = \bigcup_{1 \leq i \leq r} T_i$ is a minimum H-transversal of G. Note that the number of elements of T is $|T| = \sum_{i=1}^{r} k_i + l$ and we have $|T| > k$. Let S be any set of vertices of G such that $|S| \leq |T| - 1$ and for every i denote by S_i the set $S \cap V(G_i)$. Clearly, there exists $i_0 \in \{1, \cdots r\}$ such that $|S_{i_0}| \leq k_{i_0} = |T_{i_0}| - 1$. Then, $G_{i_0} - S_{i_0}$

contains a subgraph isomorphic to H, that is, G is exactly $(H, |T|-1)$ stable, and we have $\sum_{i=1}^{r} k_i + (r-1) = k$.
Let e be an edge of G and let G_i be the component containing e.

Claim . *$G-e$ is (H,k) stable if and only if $G_i - e$ is (H,k_i) stable.*

Proof Suppose that $G_i - e$ is (H,k_i) stable. Let U be a H-transversal of $G-e$. Set $U_i = U \cap V(G_i - e) = U \cap V(G_i)$ and for every $j \neq i$, $U_j = U \cap V(G_j)$. Since $(G_i - e) - U_i$ and each $G_j - U_j$, $j \neq i$, contain no subgraphs of $G-e$ isomorphic to H, we have for every j, $1 \leq j \leq r$, $|U_j| \geq k_j + 1$. Then, $|U| = \sum_{j=1}^{r} |U_j| \geq k+1$. Hence, for every set S of k vertices $(G-e)-S$ contains a subgraph isomorphic to H, that is, $G-e$ is (H,k) stable.
Conversely, suppose that $G_i - e$ is not (H,k_i) stable. Let T_i be a H-transversal of $(G_i - e) - T_i$ having k_i vertices. For every $j \neq i$ let T_j be a H-transversal of G_j having $k_j + 1$ vertices. The set $T = \cup_{j=1}^{r} T_j$ has k vertices and is a H-transversal of $G-e$, that is, $G-e$ is not (H,k) stable. □

Thus, G is minimal (H,k) stable if and only if for every i, $1 \leq i \leq r$, G_i is minimal (H,k_i) stable.
Note that, by replacing a minimal (H,k_i) stable component G_i by any minimal (H,k_i) stable graph G'_i (connected or not), we obtain again a minimal (H,k) stable graph. Thus, if G is minimum (H,k) stable then for every i, $1 \leq i \leq r$, G_i is minimum (H,k_i) stable.

□

Remark 2.10. Let r be an integer ≥ 2, $k_1, \cdots, k_r$ be r non negative integers and $k = \sum_{i=1}^{r} k_i + (r-1)$. If for every i, $1 \leq i \leq r$, G_i is a minimum (H,k_i) stable graph then the disjoint union $G_1 + G_2 + \cdots + G_r$ may not be a minimum (H,k) stable graph. For example, K_q is minimum $(K_q, 0)$ stable, $2K_q$ and K_{q+1} are minimal $(K_q, 1)$ stable, but since $e(2K_q) > e(K_{q+1})$, for $q \geq 4$ the graph $2K_q$ is not minimum $(K_q, 1)$ stable.

3. Minimum (K_q, k) stable graphs

In this section we are interested by (K_q, k) stable graphs with minimum size ($q \geq 3$). Recall that $stab(K_q, k) = Min\{e(G) \mid G \text{ is } (\mathrm{K_q, k}) \text{ stable}\}$.

3.1. **Some known results.**
We give here some known results about this topic.
By Remark 2.6 and Lemma 2.7 we have:

Properties 3.1. [2] *A minimal (K_q, k) stable graphs G has the following properties:*
P_1*) G is exactly (K_q, k) stable.*
P_2*) For every edge e, $G-e$ is exactly $(K_q, k-1)$ stable.*
P_3*) For every vertex v, $G-v$ is exactly $(K_q, k-1)$ stable.*
P_4*) Every vertex of G belongs to some q-clique of G.*
P_5*) Every edge of G belongs to some q-clique of G.*

Remark 3.2. For any two integers $q \geq 3$ and $k \geq 1$, K_{q+k} is minimal (K_q, k) stable.

Proposition 3.3. [4] *For every integer $q \geq 4$, K_{q+1} is the unique minimum $(K_q, 1)$ stable graph.*

Proposition 3.4. [4] *For every integer $q \geq 4$, K_{q+2} is the unique minimum $(K_q, 2)$ stable graph.*

Proposition 3.5. [4] *For every integer $q \geq 5$, K_{q+3} is the unique minimum $(K_q, 3)$ stable graph.*

Theorem 3.6. [2] *For every $k \geq 1$, $stab(K_3, k) = 3k+3$ and $stab(K_4, k) = 5k+5$.*

Theorem 3.7. [4] *Let G be a minimum (K_3, k) stable graph, with $k \geq 0$. Then G is isomorphic to $sK_4 + tK_3$, for any choice of s and t such that $2s + t = k + 1$.*

Theorem 3.8. [4] *Let G be a minimum (K_4, k) stable graph, with $k \geq 1$. Then G is isomorphic to $sK_6 + tK_5$, for any choice of s and t such that $3s + 2t = k + 1$*

Theorem 3.9. [4] *For every $k \geq 5$, $stab(K_5, k) = 7k + 7$.*

Theorem 3.10. [4] *Let G be a minimum (K_5, k) stable graph, with $k \geq 5$. Then G is isomorphic to $sK_8 + tK_7$, for any choice of s and t such that $4s + 3t = k + 1$*

Dudek et al. [2] defined the family $\mathcal{A}_r^{(K_q,k)}$ with $k \geq 0$, $q \geq 3$, $1 \leq r \leq k+1$ as the family of graphs consisting of r complete graphs K_{i_j} with $i_1 \geq \cdots \geq i_r \geq q$ satisfying the condition $\sum_{i=1}^{r}(i_j - q) + (r-1) = k$ and they prove that every graph in $\mathcal{A}_r^{(K_q,k)}$ is minimal (K_q, k) stable. We observe that if G is a (K_q, k) stable graph disjoint union of $r \geq 1$ cliques K_{i_j}, $1 \leq j \leq r$, then by Theorem 2.9, $G \in \mathcal{A}_r^{(K_q,k)}$. They defined a graph $G \in \mathcal{A}_r^{(K_q,k)}$ as a *balanced union* if $|i_j - i_l| \in \{0, 1\}$ for every j and l in $\{1, 2, \cdots, r\}$ and they proved that given q, k and r there is exactly one balanced union $\mathcal{B}_r^{(K_q,k)}$ in $\mathcal{A}_r^{(K_q,k)}$, and that $\mathcal{B}_r^{(K_q,k)}$ has the minimum number of edges among the graphs in $\mathcal{A}_r^{(K_q,k)}$.
In [2] the following lemma has been given. We give its proof for completeness.

Lemma 3.11. [2] *Let G_0 be a (K_q, k_0) stable graph ($k_0 \geq 0$) which has the minimum size among all graphs beeing a disjoint union of r cliques ($r \geq 1$) $G_i \equiv K_{q+k_j}$ with $1 \leq j \leq r$, $k_j \geq 0$. There exists a positive integer s and a nonnegative integer k such that*
$k \geq 0, s \leq r$, $G_0 = sK_{q+k} + (r-s)K_{q+k-1}$ with $rk + s = k_0 + 1$ and

$$e(G_0) = \frac{(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s))}{2r}.$$

Proof Suppose, without loss of generality, that $k_1 \geq k_2 \geq \cdots \geq k_r$ and that there exist two components G_i and G_j with $i < j$ such that $k_i - k_j \geq 2$. By substituting $G'_i \equiv K_{q+k_i-1}$ for G_i and $G'_j \equiv K_{q+k_j+1}$ for G_j, we obtain a new (K_q, k) stable graph G'_0 such that $e(G'_0) = e(G_0) - (k_i - k_j - 1) < e(G_0)$, which is a contradiction. Thus, for any i and any j, $0 \leq |k_i - k_j| \leq 1$. Hence, either for any i and any j k_i and k_j have the same value k and we have $G_0 = rK_{q+k}$ with $k \geq 0$, or there exist distinct k_i and k_j and we have $G_0 = sK_{q+k} + (r-s)K_{q+k-1}$ with $k \geq 1$ and $1 \leq s \leq r-1$.

If $G_0 = sK_{q+k} + (r-s)K_{q+k-1}$ then a minimum K_q-transversal of G_0 has $k_0 + 1 = s(k+1) + (r-s)k = s + rk$ vertices. Note that r divides $k_0 + 1 - s$. We have $2e(G_0) = s(q+k)(q+k-1) + (r-s)(q+k-1)(q+k-2)$. Since $k = \frac{k_0+1-s}{r}$, we obtain $e(G_0) = \frac{(r(q-1)+k_0+1-s)(r(q-2)+k_0+1+s))}{2r}$. □

3.2. **Minimum (K_q, k) stable graph for small k.** Recall that for every integer $q \geq 2$, K_q is the unique minimum $(K_q, 0)$ stable graph.

Remark 3.12. By Propositions 3.3 and 3.4, K_5 is the unique minimum $(K_4, 1)$ stable graph, K_6 is the unique minimum $(K_4, 2)$ stable graph and, by Theorem 3.8, $2K_5$ is the unique minimum $(K_4, 3)$ stable graph.

Remark 3.13. By Propositions 3.3, 3.4 and 3.5, for every $k \in \{1, 2, 3\}$ the graph K_{5+k} is the unique minimum (K_5, k) stable graph.

Lemma 3.14. [3] *K_9 and $K_6 + K_7$ are the only minimum $(K_5, 4)$ stable graphs.*

Theorem 3.15. [3] *Let G be a minimum (K_q, k) stable graph, where $q \geq 6$ and $k \leq \frac{q}{2} + 1$. Then G is isomorphic to K_{q+k}.*

Recall that for every integer $q \geq 4$, $\kappa(q)$ is the greatest integer such that for $1 \leq k < \kappa(q)$ the only minimum (K_q, k) stable graph is K_{q+k}. Then, either $e(K_{q+\kappa(q)}) > stab(K_q, \kappa(q))$ or $e(K_{q+\kappa(q)}) = stab(K_q, \kappa(q))$ but there is a minimum $(K_q, \kappa(q))$ stable graph G such that $K_{k+\kappa(q)} \not\equiv G$.

Lemma 3.16. *$\kappa(4) = 3$ and $\kappa(5) = 4$.*

Proof By Remark 3.12, $\kappa(4) = 3$ (and $2K_5$ is the unique minimum $(K_4, 3)$ stable graph). By Remark 3.13 and by Lemma 3.14, $\kappa(5) = 4$ (and K_9 and $K_6 + K_7$ are the minimum $(K_5, 4)$ stable graphs). $\square$

In the following, if no confusion is possible, we simply denote the integer $\kappa(q)$ by κ.

Lemma 3.17. *Suppose that $q \geq 4$. If κ is even then $stab(K_q, \kappa - 1) < e(2K_{q+\frac{\kappa}{2}-1})$ and $stab(K_q, \kappa) \leq e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})$.*
If κ is odd then $stab(K_q, \kappa - 1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$ and $stab(K_q, \kappa) \leq e(2K_{q+\frac{\kappa-1}{2}})$.

Proof Recall that, by definition of κ, $K_{q+\kappa-1}$ is the only one minimum $(K_q, \kappa - 1)$ stable. If κ is even then $2K_{q+\frac{\kappa}{2}-1}$ is exactly $(K_q, \kappa - 1)$ stable and $K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}$ is exactly (K_q, κ) stable. If κ is odd then $K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}$ is exactly $(K_q, \kappa - 1)$ stable and $2K_{q+\frac{\kappa-1}{2}}$ is exactly (K_q, κ) stable. $\square$

Lemma 3.18. *Let $q \geq 3$ and $p \geq 0$ be two integers. Then,*
$e(K_{q+2p}) < e(K_{q+p} + K_{q+p-1})$ if and only if $p^2 + p < \frac{1}{2}(q-1)(q-2)$ and
$e(K_{q+2p}) = e(K_{q+p} + K_{q+p-1})$ if and only if $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ is an integer and $p = p_0$.
$e(K_{q+2p+1}) < e(2K_{q+p})$ if and only if $(p+1)^2 < \frac{1}{2}(q-1)(q-2)$ and
$e(K_{q+2p+1}) = e(2K_{q+p})$ if and only if $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ is an integer and $p = p_1$.

Proof It is easy to check that $e(K_{q+2p}) - e(K_{q+p} + K_{q+p-1}) = p^2 + p - \frac{1}{2}(q-1)(q-2)$ and $e(K_{q+2p+1}) - e(2K_{q+p}) = (p+1)^2 - \frac{1}{2}(q-1)(q-2)$. These polynomials of degree 2 in p have respectively $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ and $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ as positive roots. $\square$

Proof of Theorem 1.6. If $\kappa = 2p$ then, by Lemma 3.17, $stab(K_q, \kappa - 1) < e(2K_{q+\frac{\kappa}{2}-1})$. Since $\kappa - 1 = 2(p-1) + 1$, by Lemma 3.18, $p^2 < \frac{1}{2}(q-1)(q-2)$, that is, $\kappa < \sqrt{2(q-1)(q-2)}$.

If $\kappa = 2p+1$ then by Lemma 3.17, $stab(K_q, \kappa-1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$. Since $\kappa - 1 = 2p$, by Lemma 3.18, $p < \frac{1}{2}(\sqrt{1+2(q-1)(q-2)} - 1)$, that is, $\kappa < \sqrt{1+2(q-1)(q-2)}$. $\square$

Lemma 3.19. *For every integer $q \geq 4$ and $\kappa = \kappa(q)$ we have $e((\kappa+1)K_q) > e(K_{q+\kappa})$*

Proof We have $2(e((\kappa+1)K_q) - e(K_{q+\kappa})) = \kappa(q^2-3q-\kappa+1)$. By Theorem 1.6, $\kappa < \sqrt{1+2(q-1)(q-2)} < \frac{3q}{2}$, and hence $\kappa(q^2-3q-\kappa+1) > 0$. $\square$

Theorem 3.20. *Let $q \geq 4$ and suppose that there exists a minimum (K_q, κ) stable graph G_0 which is disconnected. Then G_0 is isomorphic to $K_{q+\lfloor\frac{\kappa}{2}\rfloor} + K_{q+\lfloor\frac{\kappa-1}{2}\rfloor}$.*

Proof Let G_0 be a minimum (K_q, κ) stable disconnected graph having $r \geq 2$ connected components $G_1, G_2, \cdots, G_r$. By Theorem 2.9, there are integers $k_1 \geq k_2 \geq \cdots \geq k_r$ with $\sum_{i=1}^{r} k_i + (r-1) = \kappa$ such that for $1 \leq i \leq r$, G_i is minimum (K_q, k_i) stable. For every i, since $k_i < \kappa$, we have $G_i \equiv K_{q+k_i}$.
We shall prove first that $r = 2$. In fact, it is clear that $r \geq 2$. Let us suppose that $r \geq 3$. We have $k_r + k_{r-1} = \kappa - (k_{r-2} + k_{r-3} + \dots + k_1) - (r-1) \leq \kappa - 2$. Hence, $e(K_{q+k_r+k_{r-1}+1}) < e(K_{q+k_r}) + e(K_{q+k_{r-1}})$ and the graph $K_{q+k_1} + K_{q+k_2} + \cdots + K_{q+k_{r-2}} + K_{q+k_{r-1}+k_r+1}$ is (K_q, κ) stable with strictly smaller size than $K_{k_1} + K_{k_2} + \cdots + K_{k_r}$, a contradiction. So, $G_0 \in \mathcal{B}_2^{(K_q,\kappa)}$ and the proof follows. $\square$

Note that Theorem 3.20 implies that there exists at most one disconnected (K_q, κ) stable graph and this graph, if it exists, is

- either isomorphic to $K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}$ (if κ is even)
- or else isomorphic to $2K_{q+\frac{\kappa-1}{2}}$ (if κ is odd).

Proof of Theorem 1.7 By Lemma 3.17 and Theorem 3.20,
if κ is odd then

$$e(K_{q+\kappa-1}) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}) < stab(K_q, \kappa) = e(2K_{q+\frac{\kappa-1}{2}}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.18, it may be possible that $e(2K_{q+\frac{\kappa-1}{2}}) = e(K_{q+\kappa})$ for some values of q),

if κ is even then

$$e(K_{q+\kappa-1}) < e(2K_{q+\frac{\kappa}{2}-1}) < stab(K_q, \kappa) = e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.18, it may be possible that $e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) = e(K_{q+\kappa})$ for some values of q).

For $\kappa = 2p+1$ we have

$$\frac{1}{2}(q+2p)(q+2p-1) < (q+p-1)^2 < (q+p)(q+p-1) \leq \frac{1}{2}(q+2p+1)(q+2p) .$$

This implies that

$$(A) \quad p^2 + p < \frac{1}{2}(q-1)(q-2) \leq (p+1)^2 .$$

For $\kappa = 2p$ we have
$\frac{1}{2}(q+2p-1)(q+2p-2) < (q+p-1)(q+p-2) < (q+p-1)^2 \leq \frac{1}{2}(q+2p)(q+2p-1)$.
This implies that

$$(B) \quad 2p^2 < \frac{1}{2}(q-1)(q-2) \leq p^2 + p \ .$$

Combining (A) and (B) yields

$$p^2 < \frac{1}{2}(q-1)(q-2) \leq (p+1)^2 \ .$$

This implies that

$$\sqrt{\frac{1}{2}(q-1)(q-2)} - 1 \leq p < \sqrt{\frac{1}{2}(q-1)(q-2)} \ .$$

Hence, $p = \rho(q) = \lceil \sqrt{\frac{1}{2}(q-1)(q-2)}\ \rceil - 1$.
By inequalities (A) and (B), position of $\frac{1}{2}(q-1)(q-2)$ in comparison to $\rho(q)^2 + \rho(q)$ determines the parity of κ. Hence, if $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$ then $\kappa = 2\rho(q) + 1 = 2\lceil \sqrt{\frac{1}{2}(q-1)(q-2)}\ \rceil - 1$ else $\kappa = 2\rho(q) = 2\lceil \sqrt{\frac{1}{2}(q-1)(q-2)}\ \rceil - 2$ □

If there is no minimum disconnected $(K_q, \kappa(q))$ stable graph then, by definition of $\kappa(q)$, there exists a connected minimum $(K_q, \kappa(q))$ stable graph G_q distinct from a clique. Note that if such a graph exists then

$$e(G_q) \leq Min\{e(K_{q+\kappa(q)}), e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})\} \text{ if } \kappa(q) \text{ is even}$$

or

$$e(G_q) \leq Min\{e(K_{q+\kappa(q)}), e(2K_{q+\frac{\kappa-1}{2}})\} \text{ if } \kappa(q) \text{ is odd .}$$

Conjecture 1.8 states that there is no such graph G_q.

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